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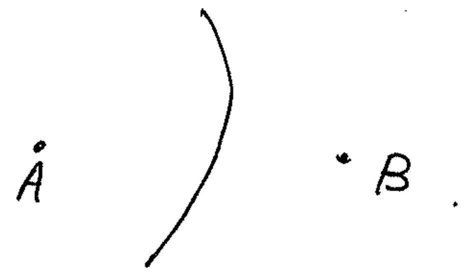
Reflection law and conformal maps

Let us again take a hard look at the (RL) that we want



Γ (RL). $u(A) + \lambda(A)\, u(\underset{\overset{\shortparallel}{B}}{R(A)}) = 0$

for all u harmonic and vanishing on Γ.
In two dimensions, with Γ given in terms of the Schwarz function $\bar{z} = S(z)$, or $z = R(z)$, $R = \overline{S}$ an anti-analytic involution, and $\lambda \equiv 1$.

In particular, it is obvious that if we have a generic (RL) for "many points" the transformation

$T_0 : u \longmapsto (T_0 u)(x) := \lambda(x)\, u(R(x))$,

where $\underbrace{}_{\text{ideally}}$ $R \circ R = id$, $R|_\Gamma = id$, T_0 must preserve harmonic functions vanishing on Γ.

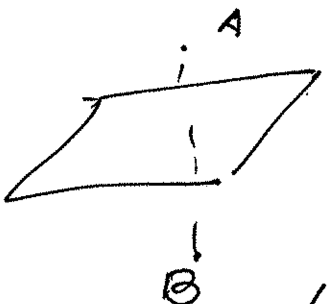
If we ask a little more, that

$$T : u(x) \longmapsto \lambda(x)\, u(R(x))$$

preserves <u>all</u> harmonic function, then (K- Shapiro, '90) R must be a conformal (or, anti-conformal) map, so in R^n, $n \geq 3$ R is either an orthogonal transformation or inversion in a sphere (in view of the Liouville theorem, so generically there are only two surfaces Γ = plane, say $x_n = 0$, or a sphere, say $\{|x| = 1\}$ with respect to which (RL) holds for "all" points near these surfaces, and it does, and well-known

$$(RL_p) \quad u(x_1, \ldots, x_{n-1}, x_n) + u(x_1, \ldots, x_{n-1}, -x_n) = 0,$$

$$u(x_1, \ldots, x_{n-1}, 0) = 0, \quad \Delta u = 0$$



Note that $\lambda \equiv 1$ here.

The second (RL_s) in the sphere is more involved

$$(RL_s) \quad u(x) + |x|^{2-n}\, u\left(\frac{x}{|x|^2}\right) \equiv 0$$

for all u: $\Delta u = 0$, $u|_{S^{n-1}} = 0$.

1c).

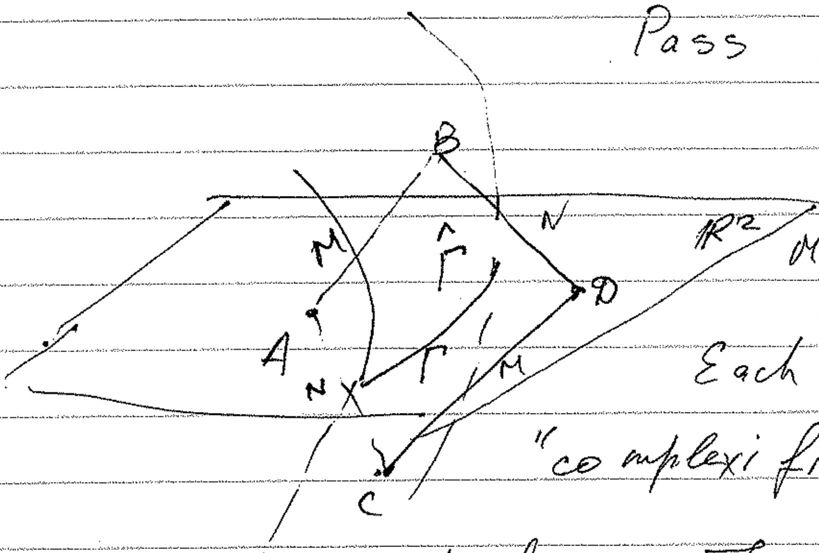
Yet, since first of all $u|_\Gamma = 0$ is a much smaller set of functions than all harmonic function, and also since one can a priori imagine that reflection holds not for "all" points A close to Γ but only for very special pairs of points A, B near Γ, the above observation doesnot answer the question in depth.

is complete

(IV) Reflection of singularities.

Let's again analyze the construction involved in the SRP.

We start with point $A \subset \mathbb{R}^2$, $A(\overset{z}{0}, \overset{\bar{z}}{0})$



Pass in $\mathbb{C}^2(z,w)$ through A two complex lines: $M\{z = z_0\}$, $N\{w = \bar{z}_0\}$

Each line intersects the "complexified" curve $\hat{\Gamma}$ at B, C respectively. Then we pass through B, C lines of "opposite" nature which meet at $D \subset \mathbb{R}^2$, $D = R(A)$. What is $M \cup N = \{(z, \bar{z}_0):$

$$(z = z_0, w = \bar{z}_0)\} = K_A\{(z - z_0)(w - \bar{z}_0) = 0\} =$$

$$= \{(x, y) \in \mathbb{C}^2 : (x - x_0)^2 + (y - y_0)^2 = 0, z_0 = x_0 + iy_0\}.$$

It is so-called "isotropic cone" in $\mathbb{C}^2$ with the vertex at A.

Hence, the involution $R : A \to \mathcal{D}$ (given by $R(z) = \overline{S(z)}$, S is the Schwarz function of the curve Γ) can be described purely geometrically

$\mathcal{D} = R(A)$ iff (3) $K_A \cap \hat{\Gamma} = K_{\mathcal{D}} \cap \hat{\Gamma} = K_A \cap K_{\mathcal{D}}$

Thus, the SRP is possible in two dimensions because near Γ (or, even $\hat{\Gamma}$) every point A ~~then~~ is in such involutive relation to another point $\mathcal{D}$ - so (3) holds.

Note that for any worthy (RL) to hold this is necessary. Indeed, let $\Gamma \subset \mathbb{R}^n, n \geq 2$ be a hypersurface, analytic ($\Gamma = \{x : \varphi(x) = 0\}$, φ is real-analytic), $\hat{\Gamma} = \{z : \varphi(z) = 0\}$ is its complexification, $\Gamma = \hat{\Gamma} \cap \mathbb{R}^n$. Suppose for all u harmonic in a neighborhood of Γ and vanishing on Γ

we have

$$(RL) \qquad u(x) + \lambda(x)\, u(R(x)) = 0$$

where $\lambda : \mathbb{R}^n \to \mathbb{R}$ is a scalar multiple.

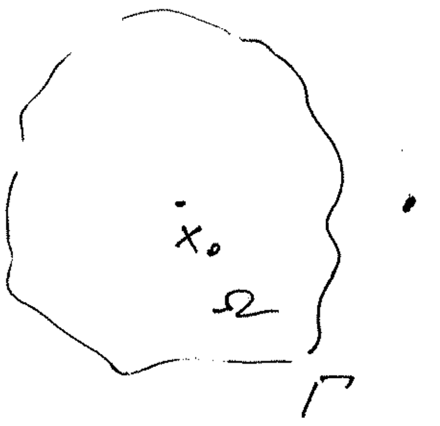
Ex. $n \geq 2$. $\Gamma := \{ \sum_1^n x_i^2 = 1 \}$. (RL) has the form

$$u(x) + |x|^{2-n}\, u\left(\frac{x}{|x|^2}\right) = 0.$$

(Remark that $x \mapsto \frac{x}{|x|^2}$ is the inversion about the unit sphere).

Then, it is natural to assume that when $u \uparrow \infty$, i.e. has a "pole" at x_0 (i.e.,

$$u(x) = \begin{cases} \frac{c}{|x-x_0|^{n-2}} + \text{regular}, & n \geq 3 \\ c \log \frac{1}{|x-x_0|} + \text{regular} & n = 2 \end{cases}$$

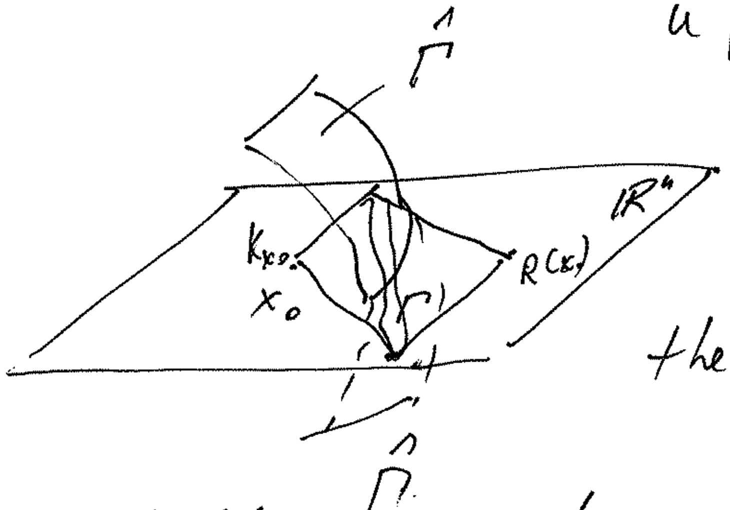


near x_0, e.g. u = Green's function of some Ω, $\partial\Omega \supset \Gamma$, with the pole at x_0), then

u must also have a pole at $R(x_0)$.

Yet, in $\mathbb{C}^n$, u is infinite not only at x_0 but on the whole isotropic

cone $K_{x_0} = \{ z \in \mathbb{C}^n : \sum_1^n (z_i - x_0^i)^2 = 0 \}$
with the vertex at x_0. At the same time
$u|_{\hat{\Gamma}} = 0$ as well as
$u|_{\Gamma} = 0$.



Thus, in order that
the singularities at x_0
and $R(x_0)$ cancel each other, we must
have the (Study) involution hold near Γ

(SR). $K_{x_0} \cap \hat{\Gamma} = K_{R(x_0)} \cap \hat{\Gamma} = K_{x_0} \cap K_{R(x_0)}$.

In '90 (~~D~~K - H. Shapiro) showed that
If (SR) holds for all points near
an <u>algebraic</u> surface $\Gamma < \mathbb{R}^n$, $n \geq 3$, Γ
must be either a plane, or a sphere.

(V) Separation of Points and (RL).

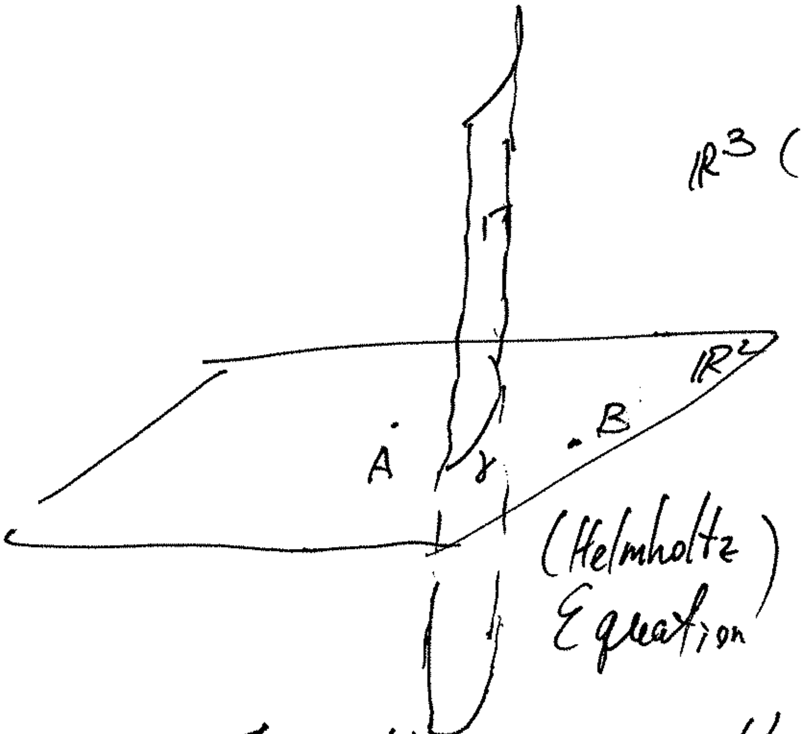
It is rather obvious that if for
any pair of points near

the hypersurface $\Gamma \subset \mathbb{R}^n$, points A, B we ~~have~~ can find u : $\Delta u = 0$, $u|_\Gamma = 0$ ($u = u(A,B)$ of course) $u(A) = 0$, $u(B) = 1$, no reflection law will ever hold.

The following theorem is due to DK - H. S. Shapiro ('90).

<u>Theorem</u> Let $\gamma \subset \mathbb{R}^2$ be an analytic curve and $\Gamma = \gamma \times \mathbb{R}$ a cylindrical surface, then functions harmonic in a neighborhood of Γ and vanishing on Γ separate points on Γ.

The most non-trivial situation is of course, when two points A, B are in the same horizontal plane and are in fact symmetric wrt γ.



$\mathbb{R}^3$ (not $\mathbb{C}^3$!).

Look for

$$u = v(x,y)\, e^{\lambda z}, \quad v\big|_\gamma = 0$$

(Helmholtz Equation) $\Delta v + \lambda^2 v = 0$ near γ.

The theorem then rests on the following

<u>Ansatz</u> If the (RL)

$$u(A) + \text{const}\; u(B) = 0$$

for all u satisfying Helmholtz equation near γ and vanishing on γ, then γ is a straight line equidistant from A, B.

Thus, there is no (RL) already for an ~~sti~~ ever so slight deviation from harmonic functions even in two dimensions although the (SR) does hold for all analytic curves in R^2.

This result answers negatively so called "antenna problem": the signal at A cannot be "reconstructed" at a point B on another side of the wall γ.

D. Aberra ('01) in his thesis showed that the Ansatz holds if we replace a point-mass at B by a distribution at B i.e.

if (MRL)

$$u(A) + \sum_{1}^{N} c_j \frac{\partial^\alpha u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}\Big|_B = 0$$

for all u: $\Delta u + \lambda^2 u = 0$, $u|_\gamma = 0$

then γ is a line equidistant from A and B.

In 1993, B. Sternin, V. Shatalov and T. Savina showed that $u(A)$ can be "re-constructed" by calculating an integral involving u, grad u over the path "connecting" B to γ.

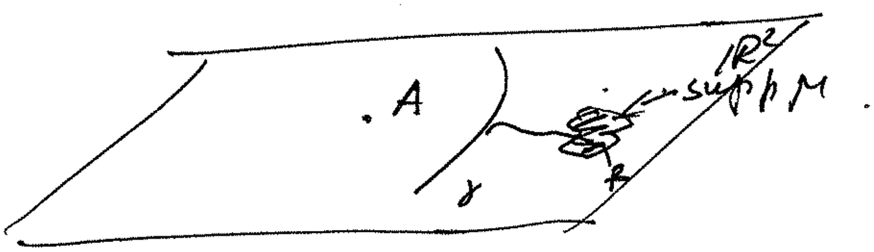
Open Problem (Generalized Antenna Problem)

Does there exist a compactly supported measure μ, $\operatorname{supp}\mu \cap \gamma = \emptyset$, s.t.

$$u(A) = \int u \, d\mu \quad \text{for all}$$

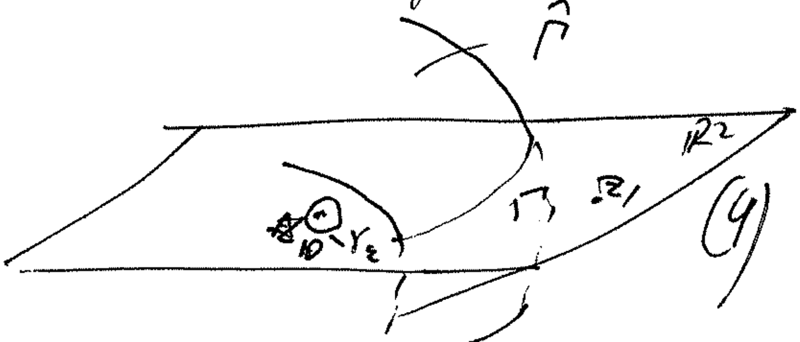
$$u\big|_{\gamma} = 0, \quad \Delta u + \lambda^2 u = 0 \quad \text{near } \gamma.$$

In other words can one "cut-off the umbilic cord" connecting B to γ.



(VI) The SRL and Huygens' Principle

Let us return to 2 dimensions and sketch yet another view point of the SRP blended from ideas of P. Garabedian and H. Lewy.



$$(4) \quad u(z^\circ) = \frac{1}{2\pi} \int_{\gamma_\varepsilon} \Big[u(\xi) \frac{\partial}{\partial n_\xi} \log|z^\circ - \xi| - \frac{\partial u}{\partial n_\xi} \log|z^\circ - \xi| \Big] ds_\xi$$