Department of Mathematics *University of South Florida*

QUALIFYING EXAM ON ALGEBRA

Saturday, January 23, 2016 from 9:00 am to 12:00 noon

Examiners: Xiang-Dong Hou and Brian Curtin

This is a three hour examination. **Write out your solutions in a clear and precise manner.** *To pass this exam at the Master's level, one should answer correctly at least 4 questions including at least one from each section. To pass this exam at the Ph.D. level, one should answer correctly at least 6 questions including two problems from at least two of the sections and at least one from each of the other two sections. Please use a new sheet of paper for each question. Please only write on one side of each page.*

Group Theory

1. A group N is said to be *complete* if the center of N is trivial and every automorphism of N is inner. Show that if G is a group, $N \lhd G$, and N is complete, then $G = N \times C_G(N)$, where $C_G(N) = \{g \in G : gn = ng \text{ for all } n \in N\}$ and $N \times C_G(N)$ is the internal direct product of N and $C_G(N)$.

2. Let p, q, r be three distinct primes, and let G be a group of order pqr. Prove that G is solvable.

3. Let A and B be two finitely generated abelian groups such that $A \oplus A \cong B \oplus B$. Prove that $A \cong B$.

Ring and Module Theory

In this section, all rings are with identity; all modules are unitary; all ring homomorphisms map identity to identity; all subrings contain the identity of the ambient ring.

1. Fix a prime number p. Let $f(x) = a_0 + a_1x + \cdots + a_kx^k + \cdots + a_nx^n \in \mathbb{Z}[x]$ such that $p \mid a_i$ for $0 \le i \le k-1$, $p \nmid a_k$, and $p^2 \nmid a_0$. Show that $f(x)$ has an irreducible factor in $\mathbb{Z}[x]$ with degree at least k.

2. Let R be a left noetherian ring and M a finitely generated left R-module. Prove that every submodule of M is finitely generated.

3. Let R be a commutative ring with 1. The Jacobson Radical $J(R)$ of R is defined to be the intersection of all maximal ideals of R. Let $x \in R$. Show that $x \in J(R)$ if and only if $1 - xy$ is a unit for all $y \in R$.

Linear Algebra

1. State and prove the Cayley-Hamilton Theorem.

2. Let P be an $n \times n$ orthogonal matrix with $\det P = -1$. Prove that $I + P$ is not invertible.

3. Let A be an 11×11 matrix over $\mathbb{C}$ whose characteristic polynomial is $x^3(x^2+1)^4$ and whose minimal polynomial is $x^2(x^2+1)^2$. Enumerate all possible Jordan canonical forms of A.

Field Theory

1. Let E be a finite Galois extension of the field F, and let $G = \mathrm{Aut}(E/F)$. Let $u \in E$. Prove that the polynomial

$$f(x) = \prod_{\sigma \in G} (x - \sigma(u))$$

has coefficients in F and is divisible by the minimal polynomial of u over F.

2. Give an example of a Galois extension L of degree 4 over $\mathbb{Q}$ such that $\mathrm{Aut}(L/\mathbb{Q}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Prove your claim.

3. Let F be a field and let $F(x)$ be the field of rational functions in one variable over F. Let

$$G = \left\{ \frac{ax+b}{cx+d} : ad - bc \neq 0 \right\} \subset F(x).$$

Then $(G, \circ)$ is a group, where $\circ$ is composition. (You do not need to prove this fact.) Prove that $\mathrm{Aut}(F(x)/F) \cong G$.

Department of Mathematics *University of South Florida*

QUALIFYING EXAM ON ALGEBRA

Saturday, May 14, 2016 from 9:00 am to 12:00 noon

Examiners: Brian Curtin and Dmytro Savchuk

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Group Theory

1. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.

2. Let G be a finite group and let $\Phi(G)$ be its Frattini subgroup, that is, the intersection of all its maximal subgroups. Show that $\Phi(G)$ is precisely the set of non-generators of G. (Recall that an element g of G is a *non-generator* if, for any subset S of G containing g and generating G, the set $S - \{g\}$ also generates G.)

3. Prove that every nilpotent group is solvable.

Ring and Module Theory

In this section, all rings are with identity; all modules are unitary; all ring homomorphisms map identity to identity; all subrings contain the identity of the ambient ring.

1. Let R be a commutative ring. Suppose that for every $a \in R$, at least one of a and $1_R - a$ is invertible. Prove that $N = \{a \in R \,|\, a \text{ is not invertible}\}$ is an ideal of R.

2. Let M be an R-module, and let N be an R-submodule of M. Prove that M is Noetherian if and only if both N and M/N are Noetherian.

3. Let I be an ideal of the commutative ring R and define

$$\text{Jac}(I) = \text{ intersection of all maximal ideals of } R \text{ that contain } I.$$

(i) Prove that $\text{Jac}(I)$ is an ideal of R containing I.

(ii) Let $n > 1$ be an integer. Describe $\text{Jac}(n\mathbb{Z})$ in terms of the prime factorization of n.

Linear Algebra

1. Two matrices A and B are *simultaneously diagonalizable* if there exists an invertible matrix P so that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal. Prove that two matrices are simultaneously diagonalizable if they commute and each is diagonalizable.

2. (i) Let A be an $n \times n$ matrix over $\mathbb{C}$ whose characteristic polynomial is $(x-\lambda)^n$. Prove that for $k \geq 1$,

$$\begin{aligned} &\operatorname{rank}(A-\lambda I)^{k-1} - \operatorname{rank}(A-\lambda I)^k \\ = &\text{the number of Jordan blocks of } A \text{ of size } l \times l \text{ with } l \geq k. \end{aligned}$$

(ii) Use (i) to determine the Jordan canonical form of

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 & 1 \\ 0 & 2 & 1 & 1 & -1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

3. Given an n-dimensional Euclidean space, two bases $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ are given so that $(v_i \,|\, v_j) = (w_i \,|\, w_j)$ for all i, j $(1 \leq i, j \leq n)$. Prove that there exists an orthogonal operator T which sends v_i to w_i $(1 \leq i \leq n)$.

Field Theory

1. Suppose $f \in K[x]$ is an irreducible polynomial of degree n and F is a field extension of K such that $[F:K] = m$ and $\gcd(n, m) = 1$. Prove that f is irreducible over F.

2. Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic extension of $\mathbb{Q}$.

3. Let F be a field with characteristic not equal to 2. Let E be a finite-dimensional Galois extension of F. Suppose that the Galois group $\operatorname{Gal}(E/F)$ is a noncyclic group of order 4. Show that $E = F(\alpha, \beta)$ for some α, $\beta \in E$ with α^2, $\beta^2 \in F$.

Department of Mathematics *University of South Florida*

QUALIFYING EXAM ON ALGEBRA

Saturday, September 24, 2016 from 9:00 am to 12:00 noon

Examiners: Brian Curtin and Dymtro Savchuk

This is a three hour examination. **Write out your solutions in a clear and precise manner.** *To pass this exam at the Master's level, one should answer correctly at least 4 questions including at least one from each section. To pass this exam at the Ph.D. level, one should answer correctly at least 6 questions including two problems from at least two of the sections and at least one from each of the other two sections. Please use a new sheet of paper for each question. Please only write on one side of each page.*

Section I: Group Theory

1. Show that if G is a finite cyclic group, then G has exactly one subgroup of order m for each positive integer m dividing $|G|$.
2. Let H and K be normal subgroups of G such that G/H and G/K are both solvable. Prove that $G/(H \cap K)$ is solvable.
3. Consider a group given by finite presentation $BS(1,m) = \langle a, b \mid a^{-1}ba = b^m \rangle$ belonging to the family of, so-called, Baumslag-Solitar groups.
   (a) Prove that each element of $BS(1,m)$ can be written in the form $a^n b^r a^{-l}$ for some $n \geq 0, l \geq 0$ and $r \in \mathbb{Z}$.
   (b) Use the result from (a) to prove that for each $m \geq 2$ the images of the generators a and b in each proper homomorphic image of $BS(1,m)$ have finite order.

Section II: Ring and Module Theory

1. Let R be a commutative ring with 1.
   (a) Show that if M is a maximal ideal of R then M is a prime ideal of R.
   (b) Give an example of a non-zero prime ideal in a ring R that is not a maximal ideal.
2. Let R be a commutative ring. Suppose $f(x) \in R[x]$ is nilpotent. Show that the coefficients of f are nilpotent.
3. Prove that if I is a nonzero ideal in a principal ideal domain R, then R/I is Artinian.

Section III: Linear Algebra

1. Let $A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{bmatrix}$

   (a) Determine conditions on a, b, c, d so that there is only one Jordan block for each eigenvalue of A in the Jordan canoncial form.

   (b) Suppose $a = c = d = 2$ and $b = -2$. Find the Jordan canonical form of A.

2. Let A, B, C, D be square matrices over some field $\mathbb{C}$. Show that if A^{-1} exists, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \cdot \det(D - CA^{-1}B).$$

3. Prove that if $A \in GL(n, \mathbb{C})$ has finite order, then A is diagonalizable over $\mathbb{C}$.

Section IV: Field Theory

1. Find the minimal polynomial of $\alpha = \sqrt{5 + \sqrt{3}}$ over the rational numbers and prove that it is the minimal polynomial.

2. Let E be a field and let G be a finite group of automorphisms of E. Let F be the fixed field of G. Prove that E is a separable algebraic extension of F.

3. Let E be a finite Galois extension of the field F, and let $G = \mathrm{Gal}(E/F)$ be the Galois group of this extension. Let $u \in E$. Prove that the polynomial

$$f(x) = \prod_{\sigma \in G} (x - \sigma(u))$$

   has coefficients in F and is divisible by the minimal polynomial of u over F.

Department of Mathematics *University of South Florida*

QUALIFYING EXAM ON ALGEBRA

Saturday, January 28, 2017 from 9:00 am to 12:00 noon

Examiners: Brian Curtin and Dmytro Savchuk

This is a three hour examination. **Write out your solutions in a clear and precise manner.** *To pass this exam at the Master's level, one should answer correctly at least 4 questions including at least one from each section. To pass this exam at the Ph.D. level, one should answer correctly at least 6 questions including two problems from at least two of the sections and at least one from each of the other two sections. Please use a new sheet of paper for each question. Please only write on one side of each page.*

Section I: Group Theory

1. Prove that a group with a subgroup of finite index also has a normal subgroup of finite index.
2. Show that the group defined by the presentation $\langle x, y, z \mid (xz)^2, (yz)^3, xyz \rangle$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ of two cyclic groups of sizes 2 and 3, respectively.
3. Let G be a group and suppose N is normal in G. Prove that G is solvable if and only if N and G/N are solvable.

Section II: Ring and Module Theory

In this section, all rings are with identity; all modules are unitary; all ring homomorphisms map identity to identity; all subrings contain the identity of the ambient ring.

1. Let R be a commutative ring and N be the set of its nilpotent elements.
   (a) Show that N is an ideal in R.
   (b) Prove that R/N is a ring with no nonzero nilpotent elements.
2. Let R be a ring and A be a simple R-module. Prove that the ring $\mathrm{Hom}_R(A, A)$ of all R-endomorphisms of A is a division ring.
3. Show that if R is a commutative Noetherian ring with identity, then the polynomial ring $R[x]$ is also Noetherian.

Section III: Linear Algebra

1. Let $A = (a_{ij}) \in M_n(\mathbb{C})$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (listed with multiplicity). Show that
$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i,j=1}^{n} a_{ij} a_{ji}.$$
In particular, if A is Hermitian, then
$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i,j=1}^{n} |a_{ij}|^2.$$

2. Let $(\,|\,)$ be the standard inner product on $\mathbb{C}^2$. Prove that there is no non-zero linear operator T on $\mathbb{C}^2$ such that $(\alpha \mid T\alpha) = 0$ for every α in $\mathbb{C}^2$.

3. Let $A = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

   (a) Find the minimal polynomial of A.

   (b) Find the Jordan canonical form of A.

Section IV: Field Theory

1. Prove that the Galois group of $x^p - 1$ over $\mathbb{Q}$ is abelian when p is prime.

2. Let K be a Galois extension of F with $|\mathrm{Gal}(K/F)| = 12$. Prove that there exists a subfield E of K containing F with $[E : F] = 3$. Does there necessarily exist a an extension L satisfying $F \subseteq L \subseteq K$ and $[L : F] = 2$? Explain.

3. Let α be a root of $x^2 - 2 \in \mathbb{F}_5[x]$ in some extension of $\mathbb{F}_5$. Thus $\mathbb{F}_5(\alpha) = \mathbb{F}_{5^2}$. (You do not need to prove this.) Prove that $\beta = \alpha + 2$ is a primitive element of $\mathbb{F}_{5^2}$, i.e., a generator of the multiplicative group $\mathbb{F}_{5^2}^*$. Then express α as a power of β.

QUALIFYING EXAM ON ALGEBRA

Saturday, May 13, 2017 from 9:00 am to 12:00 noon

Examiners: Brian Curtin and Dmytro Savchuk

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Group Theory

1. A group G is *supersolvable* if there exist normal subgroups N_i with

$$1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = G$$

such that N_{i+1}/N_i is cyclic for $0 \le i < n$. Show that a finite nilpotent group is necessarily supersolvable.

2. Show that if $|G| = p^n q$ $(n > 0)$ with $p > q$ primes, then G contains a unique normal subgroup of index q.

3. Let G be a group acting transitively on the set Ω. Prove that for each $\alpha, \beta \in \Omega$ the stabilizers G_α and G_β of α and β are isomorphic. Does the statement necessarily hold if the action of G on Ω is not transitive?

Ring and Module Theory

In this section, all rings are with identity; all modules are unitary; all ring homomorphisms map identity to identity; all subrings contain the identity of the ambient ring.

1. Let R be a noetherian UFD and suppose that whenever a, $b \in R$ are not both zero and have no common prime divisor, there exist elements u, $v \in R$ such that $au + bv = 1$. Show that R is a PID.

2. Let R be a commutative ring and M a left R-module. Prove that $\mathrm{Hom}_R(R, M)$ and M are isomorphic as left R-modules.

3. Describe all nilpotent elements in the ring $\mathbb{Z}_n$ for each $n > 1$. How many of them are there?

Linear Algebra

1. Show that a unitary operator maps every orthonormal basis onto an orthonormal basis. Does the opposite implication hold, i.e. is it true that if a linear operator maps every orthonormal basis onto an orthonormal basis, then it must be unitary?

2. Let V be an inner product space and let $u_1, u_2, \ldots, u_n$ be any n vectors in V. Show that the matrix, called *Gram matrix*,

$$G = \begin{bmatrix} \langle u_1, u_1 \rangle & \langle u_2, u_1 \rangle & \cdots & \langle u_n, u_1 \rangle \\ \langle u_1, u_2 \rangle & \langle u_2, u_2 \rangle & \cdots & \langle u_n, u_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_1, u_n \rangle & \langle u_2, u_n \rangle & \cdots & \langle u_n, u_n \rangle \end{bmatrix}$$

is positive semidefinite.

3. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & a & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a & 3 \end{bmatrix} \in M_{4\times 4}(\mathbb{C}).$$

Determine the value(s) of a such that the Jordan canonical form of A is

$$[1] \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus [3].$$

Field Theory

1. Let $E = \mathbb{Q}[i, \sqrt[8]{2}] \subseteq \mathbb{C}$. Let $F = \mathbb{Q}[i]$. Show that $\mathrm{Gal}(E/F)$ is cyclic.

2. Prove that $f(x) = x^4 + x + 1 \in \mathbb{F}_2[x]$ is irreducible. Let $\alpha \in \mathbb{F}_{2^4}$ be a root of f. Determine the multiplicative orders of α and $\alpha^2 + \alpha + 1$ respectively.

3. Prove that the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is not a finite extension of $\mathbb{Q}$.