Department of Mathematics *University of South Florida*

QUALIFYING EXAM ON ALGEBRA

Saturday, May 14, 2016 from 9:00 am to 12:00 noon

Examiners: Brian Curtin and Dmytro Savchuk

This is a three hour examination. **Write out your solutions in a clear and precise manner.** *To pass this exam at the Master's level, one should answer correctly at least 4 questions including at least one from each section. To pass this exam at the Ph.D. level, one should answer correctly at least 6 questions including two problems from at least two of the sections and at least one from each of the other two sections. Please use a new sheet of paper for each question. Please only write on one side of each page.*

Group Theory

1. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.

2. Let G be a finite group and let $\Phi(G)$ be its Frattini subgroup, that is, the intersection of all its maximal subgroups. Show that $\Phi(G)$ is precisely the set of non-generators of G. (Recall that an element g of G is a *non-generator* if, for any subset S of G containing g and generating G, the set $S - \{g\}$ also generates G.)

3. Prove that every nilpotent group is solvable.

Ring and Module Theory

In this section, all rings are with identity; all modules are unitary; all ring homomorphisms map identity to identity; all subrings contain the identity of the ambient ring.

1. Let R be a commutative ring. Suppose that for every $a \in R$, at least one of a and $1_R - a$ is invertible. Prove that $N = \{a \in R \,|\, a \text{ is not invertible}\}$ is an ideal of R.

2. Let M be an R-module, and let N be an R-submodule of M. Prove that M is Noetherian if and only if both N and M/N are Noetherian.

3. Let I be an ideal of the commutative ring R and define

$$\mathrm{Jac}(I) = \text{ intersection of all maximal ideals of } R \text{ that contain } I.$$

(i) Prove that $\mathrm{Jac}(I)$ is an ideal of R containing I.

(ii) Let $n > 1$ be an integer. Describe $\mathrm{Jac}(n\mathbb{Z})$ in terms of the prime factorization of n.

Linear Algebra

1. Two matrices A and B are *simultaneously diagonalizable* if there exists an invertible matrix P so that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal. Prove that two matrices are simultaneously diagonalizable if they commute and each is diagonalizable.

2. (i) Let A be an $n \times n$ matrix over $\mathbb{C}$ whose characteristic polynomial is $(x-\lambda)^n$. Prove that for $k \geq 1$,

$$\begin{aligned} &\operatorname{rank}(A-\lambda I)^{k-1} - \operatorname{rank}(A-\lambda I)^k \\ =&\text{the number of Jordan blocks of } A \text{ of size } l \times l \text{ with } l \geq k. \end{aligned}$$

(ii) Use (i) to determine the Jordan canonical form of

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 & 1 \\ 0 & 2 & 1 & 1 & -1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

3. Given an n-dimensional Euclidean space, two bases $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ are given so that $(v_i \,|\, v_j) = (w_i \,|\, w_j)$ for all i, j $(1 \leq i, j \leq n)$. Prove that there exists an orthogonal operator T which sends v_i to w_i $(1 \leq i \leq n)$.

Field Theory

1. Suppose $f \in K[x]$ is an irreducible polynomial of degree n and F is a field extension of K such that $[F:K] = m$ and $\gcd(n,m) = 1$. Prove that f is irreducible over F.

2. Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic extension of $\mathbb{Q}$.

3. Let F be a field with characteristic not equal to 2. Let E be a finite-dimensional Galois extension of F. Suppose that the Galois group $\operatorname{Gal}(E/F)$ is a noncyclic group of order 4. Show that $E = F(\alpha, \beta)$ for some α, $\beta \in E$ with α^2, $\beta^2 \in F$.

Department of Mathematics *University of South Florida*

QUALIFYING EXAM ON ALGEBRA

Saturday, September 24, 2016 from 9:00 am to 12:00 noon

Examiners: Brian Curtin and Dymtro Savchuk

This is a three hour examination. **Write out your solutions in a clear and precise manner.** *To pass this exam at the Master's level, one should answer correctly at least 4 questions including at least one from each section. To pass this exam at the Ph.D. level, one should answer correctly at least 6 questions including two problems from at least two of the sections and at least one from each of the other two sections. Please use a new sheet of paper for each question. Please only write on one side of each page.*

Section I: Group Theory

1. Show that if G is a finite cyclic group, then G has exactly one subgroup of order m for each positive integer m dividing $|G|$.

2. Let H and K be normal subgroups of G such that G/H and G/K are both solvable. Prove that $G/(H \cap K)$ is solvable.

3. Consider a group given by finite presentation $BS(1, m) = \langle a, b \mid a^{-1}ba = b^m \rangle$ belonging to the family of, so-called, Baumslag-Solitar groups.

   (a) Prove that each element of $BS(1, m)$ can be written in the form $a^n b^r a^{-l}$ for some $n \geq 0, l \geq 0$ and $r \in \mathbb{Z}$.

   (b) Use the result from (a) to prove that for each $m \geq 2$ the images of the generators a and b in each proper homomorphic image of $BS(1, m)$ have finite order.

Section II: Ring and Module Theory

1. Let R be a commutative ring with 1.

   (a) Show that if M is a maximal ideal of R then M is a prime ideal of R.

   (b) Give an example of a non-zero prime ideal in a ring R that is not a maximal ideal.

2. Let R be a commutative ring. Suppose $f(x) \in R[x]$ is nilpotent. Show that the coefficients of f are nilpotent.

3. Prove that if I is a nonzero ideal in a principal ideal domain R, then R/I is Artinian.

Section III: Linear Algebra

1. Let $A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{bmatrix}$

   (a) Determine conditions on a, b, c, d so that there is only one Jordan block for each eigenvalue of A in the Jordan canoncial form.

   (b) Suppose $a = c = d = 2$ and $b = -2$. Find the Jordan canonical form of A.

2. Let A, B, C, D be square matrices over some field $\mathbb{C}$. Show that if A^{-1} exists, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \cdot \det(D - CA^{-1}B).$$

3. Prove that if $A \in GL(n, \mathbb{C})$ has finite order, then A is diagonalizable over $\mathbb{C}$.

Section IV: Field Theory

1. Find the minimal polynomial of $\alpha = \sqrt{5 + \sqrt{3}}$ over the rational numbers and prove that it is the minimal polynomial.

2. Let E be a field and let G be a finite group of automorphisms of E. Let F be the fixed field of G. Prove that E is a separable algebraic extension of F.

3. Let E be a finite Galois extension of the field F, and let $G = \mathrm{Gal}(E/F)$ be the Galois group of this extension. Let $u \in E$. Prove that the polynomial

$$f(x) = \prod_{\sigma \in G} (x - \sigma(u))$$

   has coefficients in F and is divisible by the minimal polynomial of u over F.

Department of Mathematics *University of South Florida*

QUALIFYING EXAM ON ALGEBRA

Saturday, January 28, 2017 from 9:00 am to 12:00 noon

Examiners: Brian Curtin and Dmytro Savchuk

This is a three hour examination. **Write out your solutions in a clear and precise manner.** *To pass this exam at the Master's level, one should answer correctly at least 4 questions including at least one from each section. To pass this exam at the Ph.D. level, one should answer correctly at least 6 questions including two problems from at least two of the sections and at least one from each of the other two sections. Please use a new sheet of paper for each question. Please only write on one side of each page.*

Section I: Group Theory

1. Prove that a group with a subgroup of finite index also has a normal subgroup of finite index.

2. Show that the group defined by the presentation $\langle x, y, z \mid (xz)^2, (yz)^3, xyz \rangle$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ of two cyclic groups of sizes 2 and 3, respectively.

3. Let G be a group and suppose N is normal in G. Prove that G is solvable if and only if N and G/N are solvable.

Section II: Ring and Module Theory

In this section, all rings are with identity; all modules are unitary; all ring homomorphisms map identity to identity; all subrings contain the identity of the ambient ring.

1. Let R be a commutative ring and N be the set of its nilpotent elements.
   (a) Show that N is an ideal in R.
   (b) Prove that R/N is a ring with no nonzero nilpotent elements.

2. Let R be a ring and A be a simple R-module. Prove that the ring $\mathrm{Hom}_R(A, A)$ of all R-endomorphisms of A is a division ring.

3. Show that if R is a commutative Noetherian ring with identity, then the polynomial ring $R[x]$ is also Noetherian.

Section III: Linear Algebra

1. Let $A = (a_{ij}) \in M_n(\mathbb{C})$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (listed with multiplicity). Show that
$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i,j=1}^{n} a_{ij} a_{ji}.$$
In particular, if A is Hermitian, then
$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i,j=1}^{n} |a_{ij}|^2.$$

2. Let $(\ |\ )$ be the standard inner product on $\mathbb{C}^2$. Prove that there is no non-zero linear operator T on $\mathbb{C}^2$ such that $(\alpha \mid T\alpha) = 0$ for every α in $\mathbb{C}^2$.

3. Let $A = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$
   (a) Find the minimal polynomial of A.
   (b) Find the Jordan canonical form of A.

Section IV: Field Theory

1. Prove that the Galois group of $x^p - 1$ over $\mathbb{Q}$ is abelian when p is prime.

2. Let K be a Galois extension of F with $|\mathrm{Gal}(K/F)| = 12$. Prove that there exists a subfield E of K containing F with $[E : F] = 3$. Does there necessarily exist a an extension L satisfying $F \subseteq L \subseteq K$ and $[L : F] = 2$? Explain.

3. Let α be a root of $x^2 - 2 \in \mathbb{F}_5[x]$ in some extension of $\mathbb{F}_5$. Thus $\mathbb{F}_5(\alpha) = \mathbb{F}_{5^2}$. (You do not need to prove this.) Prove that $\beta = \alpha + 2$ is a primitive element of $\mathbb{F}_{5^2}$, i.e., a generator of the multiplicative group $\mathbb{F}_{5^2}^*$. Then express α as a power of β.

QUALIFYING EXAM ON ALGEBRA

Saturday, May 13, 2017 from 9:00 am to 12:00 noon

Examiners: Brian Curtin and Dmytro Savchuk

This is a three hour examination. **Write out your solutions in a clear and precise manner.** *To pass this exam at the Master's level, one should answer correctly at least 4 questions including at least one from each section. To pass this exam at the Ph.D. level, one should answer correctly at least 6 questions including two problems from at least two of the sections and at least one from each of the other two sections. Please use a new sheet of paper for each question. Please only write on one side of each page.*

Group Theory

1. A group G is *supersolvable* if there exist normal subgroups N_i with

$$1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = G$$

such that N_{i+1}/N_i is cyclic for $0 \leq i < n$. Show that a finite nilpotent group is necessarily supersolvable.

2. Show that if $|G| = p^n q$ $(n > 0)$ with $p > q$ primes, then G contains a unique normal subgroup of index q.

3. Let G be a group acting transitively on the set Ω. Prove that for each $\alpha, \beta \in \Omega$ the stabilizers G_α and G_β of α and β are isomorphic. Does the statement necessarily hold if the action of G on Ω is not transitive?

Ring and Module Theory

In this section, all rings are with identity; all modules are unitary; all ring homomorphisms map identity to identity; all subrings contain the identity of the ambient ring.

1. Let R be a noetherian UFD and suppose that whenever a, $b \in R$ are not both zero and have no common prime divisor, there exist elements u, $v \in R$ such that $au + bv = 1$. Show that R is a PID.

2. Let R be a commutative ring and M a left R-module. Prove that $\mathrm{Hom}_R(R, M)$ and M are isomorphic as left R-modules.

3. Describe all nilpotent elements in the ring $\mathbb{Z}_n$ for each $n > 1$. How many of them are there?

Linear Algebra

1. Show that a unitary operator maps every orthonormal basis onto an orthonormal basis. Does the opposite implication hold, i.e. is it true that if a linear operator maps every orthonormal basis onto an orthonormal basis, then it must be unitary?

2. Let V be an inner product space and let $u_1, u_2, \ldots, u_n$ be any n vectors in V. Show that the matrix, called *Gram matrix*,

$$G = \begin{bmatrix} \langle u_1, u_1\rangle & \langle u_2, u_1\rangle & \cdots & \langle u_n, u_1\rangle \\ \langle u_1, u_2\rangle & \langle u_2, u_2\rangle & \cdots & \langle u_n, u_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_1, u_n\rangle & \langle u_2, u_n\rangle & \cdots & \langle u_n, u_n\rangle \end{bmatrix}$$

is positive semidefinite.

3. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & a & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a & 3 \end{bmatrix} \in M_{4\times 4}(\mathbb{C}).$$

Determine the value(s) of a such that the Jordan canonical form of A is

$$[1] \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus [3].$$

Field Theory

1. Let $E = \mathbb{Q}[i, \sqrt[8]{2}] \subseteq \mathbb{C}$. Let $F = \mathbb{Q}[i]$. Show that $\mathrm{Gal}(E/F)$ is cyclic.

2. Prove that $f(x) = x^4 + x + 1 \in \mathbb{F}_2[x]$ is irreducible. Let $\alpha \in \mathbb{F}_{2^4}$ be a root of f. Determine the multiplicative orders of α and $\alpha^2 + \alpha + 1$ respectively.

3. Prove that the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is not a finite extension of $\mathbb{Q}$.

Department of Mathematics *University of South Florida*

QUALIFYING EXAM ON ALGEBRA

Saturday, September 30, 2017 from 9:00 am to 12:00 noon

Examiners: Brian Curtin and Xiang-dong Hou

This is a three hour examination. **Write out your solutions in a clear and precise manner.** *To pass this exam at the Master's level, one should answer correctly at least 4 questions including at least one from each section. To pass this exam at the Ph.D. level, one should answer correctly at least 6 questions including two problems from at least two of the sections and at least one from each of the other two sections. Please use a new sheet of paper for each question. Please only write on one side of each page.*

Group Theory

1. Let $S \in \mathrm{Syl}_p(G)$ and $N \lhd G$. Show that $S \cap N \in \mathrm{Syl}_p(N)$. In particular, if N is a p-group, then $N \subseteq S$.

2. Let G be a finite group, and let H be a non-normal subgroup of prime index p. Prove that the number of distinct conjugates of H in G is p.

3. Let N be a normal subgroup of G.
(i) Prove that G is solvable if an only if both N and G/N are solvable.
(ii) Use an example to prove that the statement in (i) is false when "solvable" is replaced with "nilpotent".

Ring and Module Theory

In this section, all rings are with identity; all modules are unitary; all ring homomorphisms map identity to identity; all subrings contain the identity of the ambient ring.

1. If R is a commutative ring with identity and $f = a_n x^n + \cdots + a_0$ is a zero divisor in $R[x]$, then there exists a nonzero $b \in R$ such that $ba_n = ba_{n-1} = \cdots = ba_0 = 0$.

2. Use the Wedderburn Theorem to show that a commutative semi-simple ring is a direct product of finitely many fields.

3. Let R be a ring, and let M be a left R-module. Suppose $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ is a chain of submodules such that for $i = 1, 2, \ldots, n$, the factors M_i/M_{i-1} are simple and pairwise non isomorphic. Prove that if X and Y are isomorphic submodules of M, then $X = Y$.

Linear Algebra

1. Prove the Rank-Nullity Theorem: Let V be a finite-dimensional vector space, and let W be a vector space over some field F. Let $T : V \to W$ be a linear map. Then $\dim V = \operatorname{rank}(T) + \operatorname{nullity}(T)$.

2. Let A be a linear operator on $\mathbb{R}^n$ and let A^* be its adjoint operator, i.e., the operator defined on the linear space of linear functionals on $\mathbb{R}^n$ by $(A^* f)(x) = f(Ax)$ for every $x \in \mathbb{R}^n$ and every linear functional f on $\mathbb{R}^n$. Show that A is invertible if and only if A^* is.

3. Let A, $B \in M_n(\mathbb{C})$ be such that $AB = BA$. Let λ be an eigenvalue of A. Prove that the eigenspace $\mathcal{E}_A(\lambda)$ is B-invariant, i.e., $B\mathcal{E}_A(\lambda) \subseteq \mathcal{E}_A(\lambda)$. Use this to show that A and B have a common eigenvector.

Field Theory

1. Let $F \subseteq K \subseteq E$, where E is a finite degree Galois extension of F. Prove that for all elements $\alpha \in E$, $N_{E/F}(\alpha) = N_{K/F}(N_{E/K}(\alpha))$. Recall that the norm of the Galois extension E/F satisfies $N_{E/F}(\alpha) = \prod_{\sigma \in \mathrm{Gal}(E/F)} \sigma(\alpha)$ for all $\alpha \in E$.

2. Let F be any field and let $F(x)$ be the field of rational functions over F. For $R(x) = A(x)/B(x) \in F(x) \setminus F$, where $A(x)$, $B(x) \in F[x]$ and $\gcd(A(x), B(x)) = 1$, define $\deg R(x) = \max\{\deg A(x), \deg B(x)\}$. Prove that $[F(x) : F(R(x))] = \deg R(x)$.

3. Suppose $E \supseteq F$ is a finite degree Galois extension and $\mathrm{Gal}(E/F)$ is isomorphic to a transitive subgroup of the symmetric group S_n. Show that E is a splitting field over F for some irreducible polynomial $f \in F[x]$ with $\deg(f) = n$.

Department of Mathematics *University of South Florida*

QUALIFYING EXAM ON ALGEBRA

Saturday, January 27, 2018 from 9:00 am to 12:00 noon

Examiners: Brian Curtin and Xiang-dong Hou

This is a three hour examination. **Write out your solutions in a clear and precise manner.** *To pass this exam at the Master's level, one should answer correctly at least 4 questions including at least one from each section. To Pass at the Ph.D. level, one should answer correctly at least 6 questions including two problems from each of the sections. Please use a new sheet of paper for each question. Please only write on one side of each page.*

Group Theory

1. Let A and B be groups such for all $a \in A$ and $b \in B$, $o(a) < \infty$, $o(b) < \infty$, and $\gcd(o(a), o(b)) = 1$. Prove that every subgroup of $A \times B$ is of the form $A_1 \times B_1$ for some $A_1 < A$ and $B_1 < B$.

2. Let G be a group of order pqr, where $p < q < r$ are primes. Prove that the Sylow r-group of G is normal.

3. Let G be a finite group with a normal subgroup N of order 3 that is not contained in the center of G. Show that G has a subgroup of index 2.

Hint: Use a group action

4. Let G be a finite nilpotent group such that $p^3 \nmid |G|$ for every prime p. Prove that G is abelian.

Rings & Modules and Fields on page 2 $\longrightarrow$

Ring and Module Theory

In this section, all rings are with identity; all modules are unitary; all ring homomorphisms map identity to identity; all subrings contain the identity of the ambient ring.

1. Let p be a prime and n be a positive integer. Show that $f(x) = \sum_{i=0}^{p-1} x^{ip^n} \in \mathbb{Q}[x]$ is irreducible.

2. Let R be a ring and let $P_1, \ldots, P_n$ be prime ideals of R. Let I be an ideal of R such that $I \subset \bigcup_{i=1}^{n} P_i$. Prove that $I \subset P_i$ for some $1 \leq i \leq n$.

3. Let R be a commutative Noetherian ring with identity. Prove that there are only finitely many minimal prime ideals of R.

4. Let F be a field and R the matrix ring defined by

$$R = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} : a_{ij} \in F \right\}.$$

Find the Jacobson radical of R. Recall that the Jacobson radical of a ring is the intersection of all maximal left ideals.

Field Theory

1. Let E be the splitting field of $f(X) = X^4 - 6X^2 + 7$ over $\mathbb{Q}$ in $\mathbb{C}$. Determine the structure of $\mathrm{Gal}(E/\mathbb{Q})$.

2. Let F/K be a finite dimensional Galois extension and E be an intermediate field. Prove that there is a unique smallest field L such that $E \subset L \subset F$ and L/K is Galois. Moreover,

$$\mathrm{Aut}(F/L) = \bigcap_{\sigma \in \mathrm{Aut}(F/K)} \sigma \mathrm{Aut}(F/E) \sigma^{-1}.$$

3. Show that every finite extension of a finite field is Galois.

4. Let $\mathbb{F}_2(x, y)$ be the field of rational functions in x, y over $\mathbb{F}_2$.
(i) Prove that $[\mathbb{F}_2(x, y) : \mathbb{F}_2(x^2, y^2)] = 4$.
(ii) Prove that there are infinitely many fields between $\mathbb{F}_2(x^2, y^2)$ and $\mathbb{F}_2(x, y)$. Conclude that $\mathbb{F}_2(x, y)$ is not a simple extension of $\mathbb{F}_2(x^2, y^2)$.