Real Analysis Qualifying Exam

Spring 2016

by B. Shekhtman & S. Kouchekian

Instructions: Do **7** of the following **9** problems. Show your work. State the theorems you use so that all the hypothesis are checked. Justify your reasoning in details.

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1. Suppose f is uniformly continues on the interval $(0, 1)$. Prove that f has left-hand side limit at 1; i.e., $\lim\limits_{x\to 1^-} f(x)$ exists.

2. Let μ be the Lebesgue measure on $\mathbb{R}^d$ and suppose $f \in L^1(\mathbb{R}^d)$. Show that if $\displaystyle\int_E |f|\, d\mu \le \mu(E)$ for all Lebesgue measurable sets $E \subseteq \mathbb{R}^d$, then $|f| \le 1$ almost everywhere.

3. Prove that the sequence $\{a_n\}_{n=1}^{\infty}$, defined by
$$a_n = \int_0^1 \Big(1 + \frac{x}{n}\Big)^{-n} x^{-\frac{1}{n}}\, dx,$$
converges and find its limit.

4. Determine all real valued continuous functions g on $[-\pi, \pi]$ such that
$$\int_{-\pi}^{\pi} g(t)\sin(nt)\, dt = 0$$
for all $n = 1, 2, ...$

5. Let $\{f_n\}_{n=0}^{\infty}$ be an orthonormal sequence of functions in $L^2[0, 1]$; i.e.,
$$\int_0^1 f_m(t) f_n(t)\, dt = \begin{cases} 0 & \text{if } \ m \ne n \\ 1 & \text{if } \ m = n. \end{cases}$$

   (a) Show that the operators $P_n : L^2[0, 1] \to L^2[0, 1]$ defined by
   $$\big(P_n f\big)(x) = \sum_{k=0}^{n} \Big(\int_0^1 f(t) f_k(t)\, dt\Big) f_k(x), \quad \text{where} \quad n = 1, 2\cdots,$$
   are uniformly bounded.

   (b) Show that $\displaystyle\lim_{n\to\infty} \int_E f_n(t)\, dt = 0$ for any measurable set $E \subseteq [0, 1]$.

6. Let f be a Lebesgue integrable, real valued function on $(0,1)$. For $x \in (0,1)$ define

$$g(x) = \int_x^1 t^{-1} f(t) dt.$$

Show that g is Lebesgue measurable and

$$\int_0^1 f(x) dx = \int_0^1 g(x) dx.$$

7. Let μ and ν be σ-finite measures on the measurable space $(X, \mathcal{M})$. Let f be a nonnegative measurable function with respect to $\mathcal{M}$. Show that if $\nu << \mu$, then

$$\int_X f \, d\nu = \int_X f \left[\frac{d\nu}{d\mu}\right] d\mu.$$

8. Recall that $C[01]$, the space of continuous real–valued functions on $[0,1]$, is a Banach space under the supremum–norm. Let

$$\mathcal{H} = \Big\{ f \in C[0,1] : f(0) = 0, f \text{ is absolutely continuous, and } f' \in L^2[0,1] \Big\}.$$

   (a) Prove that $\langle f, g \rangle = \int_0^1 f'(x) g'(x) \, dx$ defines an inner product on $\mathcal{H}$.
   (b) Prove that $\big(\mathcal{H}, \langle \cdot, \cdot \rangle\big)$ is a Hilbert space.
   (c) Show that the injection $i : \mathcal{H} \to C[0,1]$ is continuous.

9. Let X be a Banach space with closed subspaces M and N such that $M \cap N = \{0\}$ and every x in X can be written in the form $x = y + z$ for some y in M and $z \in N$. If P is defined on X by $P(y+z) = y$, prove that P is well–defined, linear, and continuous from X into X.

Real Analysis Qualifying Exam – May 14th 2016

Written by Prof. S. Lee and Prof. B. Shekhtman

Solve 8 out of 12 problems.

(1) Prove the Banach contraction principle:
Let T be a mapping from a complete metric space X into itself such that

$$d(Tx, Ty) \leq q d(x, y)$$

for all $x, y \in X$ and for some $q < 1$. For arbitrary $x_0 \in X$ let $x_n = Tx_{n-1}$. Prove that $x_n \to x$ such that $Tx = x$.(x is a fixed point for T). Also prove that this is the unique fixed point for T.

(2) Does the following limit exist

$$\lim \int_0^1 \frac{dx}{(1+\frac{x}{n})^n x^{\frac{1}{n}}}?$$

If it does, find the limit.

(3) Let f be a continuous linear functional on a Hilbert space H and $M := \{x \in H : f(x) = 0\}$. Prove that $\dim M^{\perp} \leq 1$.

(4) Let T be a surjective linear map from a Banach space X onto a Banach space Y such that

$$\|Tx\| \geq \frac{1}{2016} \|x\|$$

for all $x \in X$. Show that T is bounded.

(5) Let $A := span\{x^n(1-x) : n \geq 1\} \subset C([0,1])$. Describe its closure in the uniform norm.

(6) Let μ be a finite positive measure on the measurable space (Ω, Ξ, μ) and let $F \in L_p(\mu)^*$ for some $1 \leq p < \infty$. Prove that there exists $g \in L_1(\mu)$ such that

$$\int_A g d\mu = F(\chi_A)$$

for all $A \in \Xi$. (Here χ_A is the characteristic function of A).

(7) Show that

$$f \in L^1(\mathbb{R}) \quad \Longleftrightarrow \quad g \in L^1(\mathbb{R} \times \mathbb{R})$$

where $g(x, y) = f(x+y)f(x)$.

(8) Show that, for $1 \leq p < \infty$, a bounded sequence $\{f_n\}$ in $L^p(\mathbb{R})$ such that $\{f_n\} \to f$ pointwise a.e. converges weakly to f in $L^p(\mathbb{R})$.

(9) Let λ, μ be finite measures on X. Let

$$F = \{f \in L^1(X, \mu) : \int_E f d\mu \leq \lambda(E)\}$$

for all measurable sets E. Show that there exists $f_0 \in F$ such that

$$\int_X f_0 d\mu = \sup_{f \in F} \int_X f d\mu.$$

(10) Find a bounded sequence in $L^1([0,1])$ such that

$$\lim_{n\to\infty} \int_0^x f_n = \int_0^x f \qquad \text{for all } x \in [0,1]$$

and $\{f_n\}$ does not converge weakly to f in $L^1([0,1])$.

(11) Let f be Lipschitz on $\mathbb{R}$ and g be absolutely continuous on $[0,1]$. Show that the composition $f \circ g$ is absolutely continuous.

(12) Describe the Carathéodory construction of a measure from a set function $\mu : S \to [0, \infty]$ where S is a collection of subsets in X. Show that μ is countably monotone if and only if the outer measure induced by μ is an extension of μ.

Real Analysis Qualifying Exam – September 24th 2016

Written by S. Kouchekian and S. Lee

INSTRUCTIONS : Do at least 7 problems. Justify your reasoning. State the theorems you use so that all the hypothesis are checked.

1. Suppose that f is a measurable function in $\mathbb{R}$. Prove that there exists a sequence of step functions that converges pointwise to $f(x)$ for almost every x.

2. Let $F : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$F(x) = \int_a^x f(y)dy$$

for an integrable function. Prove that F is absolutely continuous.

3. Compute the following limit and justify the calculation.

$$\sum_{n=0}^{\infty} \int_0^{\pi/4} (1 - \sqrt{\sin x})^n \cos x \ dx.$$

4. Let $(X, \mathcal{M}, \mu)$ be a σ-finite positive measure space and $\{\nu_n\}$ be a sequence of σ-finite positive measures on $\mathcal{M}$ with $\nu_n \ll \mu$ for all $n \in \mathbb{N}$ such that

$$\nu_n(E) \leq \nu_{n+1}(E) \quad \text{for all } n \in \mathbb{N}, \quad E \in \mathcal{M}.$$

Define $\nu(E) = \lim_{n\to\infty} \nu_n(E)$ for each $E \in \mathcal{M}$. Show that ν defines a measure on $\mathcal{M}$ and ν satisfies $\nu \ll \mu$. And express $\frac{d\nu}{d\mu}$ in terms of ν_n.

5. Let $f_n, f \in L^p[0,1]$ with $1 \leq p < \infty$. Suppose that $f_n \to f$ pointwise. Prove that $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$.

6. For each polynomial f on $[0,1]$, let

$$\|f\| = \|f\|_\infty + \|f'\|_\infty.$$

Let X be the normed linear space of polynomials on $[0,1]$ with the norm $\|\cdot\|_\infty$ and Y be the normed linear space of polynomials on $[0,1]$ with the norm $\|\cdot\|$. Let $T : X \to Y$ be the linear operator defined by $T(f) = f'$ for $f \in X$. Show that T is unbounded.

The graph of T is the set $\{(x, T(x)) \in X \times Y | x \in X\}$. Is the graph of T closed in the product topology of $X \times Y$?

7. Let $(X, \mathcal{M}, \mu)$ be a σ-finite positive measure space and ν be a finite measure on $\mathcal{M}$. Show that $\nu \perp \mu$ if and only if there is no nonzero measure ρ on $\mathcal{M}$ such that $\rho \ll \mu$ and $\rho \leq \nu$ on $\mathcal{M}$.

8. Given an example of an increasing function on $\mathbb{R}$ whose set of discontinuity is precisely $\mathbb{Q}$.

9. Let $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ with $1 \leq p < q < \infty$. Prove that $f \in L^r(\mathbb{R})$ for any r with $p < r < q$.

Real Analysis Qualifying Exam – January 28th 2017

Written by B. Shekhtman and S. Lee

INSTRUCTIONS : Do at least 7 problems. Justify your reasoning. State the theorems you use so that all the hypothesis are checked.

1. Suppose that f and g are continuous function on $[0,1]$ and

$$\int_0^1 f(x)x^n dx = \int_0^1 g(x)x^n dx$$

for all $n \geq 0$. Prove that $f(x) = g(x)$.

2. Let X be a Banach space and Y be a proper closed subspace of X. Prove that for every $0 < \varepsilon < 1$ there exists $x \in X$ with $\|x\| = 1$ such that

$$\|x - y\| > \varepsilon$$

for all $y \in Y$.

3. Let T be a bounded linear operator on a Hilbert space H. Show that $\|T\| = \|T^*\|$ and $\|T^*T\| = \|T\|^2$.

4. Let f be a non-negative function in $L_1([0,1])$. Show that

$$\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}dx = m\{x \in [0,1] : f(x) > 0\}.$$

(Here m stands for Lebegues measure on $[0,1]$).

5. Let f be a continuous function on $[0,1]$. Show that $\lim_{n\to\infty} \int_0^1 (n+1)x^n f(x)dx = f(1)$.

6. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f \in L^p(\mu) \cap L^\infty(\mu)$. Show that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$

7. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of nonnegative measurable functions on $[0,1]$ with

$$\int_0^1 f_n dx \leq \frac{1}{n^2} \quad \text{for all } n \geq 1.$$

Prove that $f_n \to 0$ a.e. on $[0,1]$.

8. Let $f \in L_1(\mathbb{R})$. Show that $g(x) = \int_{-\infty}^{\infty} \cos(xt) f(t) dt$ is continuous and bounded.

9. Let $\{E_n\}$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ with

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty. \tag{1}$$

a) Show that

$$\mu(\limsup_{n\to\infty} E_n) = 0,$$

where $\limsup_{n\to\infty} E_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n$

b) Is the conclusion in a) still true if (1) is replaced by $\sum_{n=1}^{\infty} (\mu(E_n))^2 < \infty$?

10. Let $0 < a < b$. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} e^{-xy} & (x,y) \in [0,\infty) \times [a,b], \\ 0 & \text{otherwise} \end{cases}$$

is integrable with respect to the Lebesgue measure on $\mathbb{R}^2$. Compute the integral

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx.$$

11. Prove $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous over any compact set if and only $f \cdot g$ is absoletely continuous for all smooth $g : \mathbb{R} \to \mathbb{R}$ supported on a compact set.

12. Let $E \subset [0,1]$ be measurable. Define a function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \int_0^x (1 - 2\chi_E(t)) dt.$$

Prove that f is of bounded variation and determine the total variation of f on $[0,1]$.

QUALIFYING EXAM: REAL ANALYSIS
May 13, 2017

Prof. C. Bénéteau

Prof. S.-Y. Lee

Answer 4 questions from Part A and 3 questions from Part B.

Part A.

1. Let m be Lebesgue measure on $\mathbb{R}$, and suppose that $E \subset [0,1]$ is a set of real numbers with the property that for any x and y in E with $x \neq y$, $x - y$ is not equal to a rational number. Prove that the collection $\{E + r : r \in \mathbb{Q} \cap [4,5]\}$ is a countable collection of mutually disjoint sets. Prove that either $m(E) = 0$ or E is not measurable.

2. Let $X = [0,1]$ and let m be Lebesgue measure on X. Define what it means for a sequence of measurable functions f_n to converge to a function f in measure. Prove or disprove the following statement: If f_n converges to f in measure, then f_n converges to f pointwise a.e.

3. Consider the function $f(x) = x\cos(1/\sqrt{x})$ defined on the interval $[0,1]$, where we set $f(0) = 0$. Is f of bounded variation? Prove or disprove.

4. (a) Given a measure space $(X, \mathcal{M}, \mu)$ and a sequence of integrable functions $\{f_n\}_{n=1}^{\infty}$, define what it means for the sequence to be uniformly integrable.

   (b) If $X = [0,\infty)$, $\mathcal{M}$ is all Lebesgue measurable sets, and μ is Lebesgue measure, is the sequence $f_n(x) = \sqrt{n}\,\chi_{[0,1/n)}(x)$ uniformly integrable? Prove or disprove.

5. (a) State the Radon-Nikodym Theorem.

   (b) Give an example that shows the necessity of the requirement that the space X be σ finite in the Radon-Nikodym theorem.

   (c) Suppose $X = [0,1]$, $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets, μ is Lebesgue measure on X, and ν is counting measure on X. Prove or disprove each of the following statements: (1) $\mu << \nu$; (2) $\nu << \mu$; (3) μ and ν are mutually singular.

6. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of nonnegative measurable functions on $[0,1]$ with
$$\int_0^1 f_n\,dx \leq a_n \quad \text{for all } n \geq 1,$$
where $\sum_{n=1}^{\infty} a_n < \infty$. Prove that $f_n \to 0$ a.e. on $[0,1]$.

7. Let $F : \mathbb{R} \to \mathbb{R}$ be a function satisfying
$$F(x) = \int_a^x f(y)dy$$
for an integrable function f. Prove (without referring to any theorems) that F is absolutely continuous.

Part B.

8. Let $f_n(\theta) = e^{in\theta}$ for $0 \le \theta \le 2\pi$. Show that $f_n(\theta) \to 0$ weakly in $L^p[0, 2\pi]$ for any $1 < p < \infty$.

9. Let μ be a complex measure μ on a measure space $(X, \mathcal{M})$, i.e., μ is a countably additive set function from $\mathcal{M}$ to $\mathbb{C}$ satisfying $\mu(\emptyset) = 0$. Define a new set function $|\mu|$ on $\mathcal{M}$ as follows:
$$|\mu|(E) := \sup \left\{ \sum_{k=1}^{\infty} |\mu(E_k)| : E_k \in \mathcal{M} \text{ are pairwise disjoint }, E = \cup_{k=1}^{\infty} E_k \right\}.$$
Prove that $|\mu|$ is a positive measure. ($|\mu|$ is sometimes called the total variation measure.)

10. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Show that $L^p(X, \mu) \subsetneq L^1(X, \mu)$ for any $p > 1$. Is it true if we remove the hypothesis that $\mu(X) < \infty$?

11. Let $X = Y = [0, 1]$ and for each positive integer n, let φ_n be the characteristic function of the interval $[1 - 1/2^{n-1}, 1 - 1/2^n)$, and let $g_n = 2^n \varphi_n$. Define
$$f(x, y) = \sum_{n=1}^{\infty} [g_n(x) - g_{n+1}(x)]\, g_n(y).$$
Prove that f is measurable as a function on the product space $X \times Y$ (with respect to Lebesgue measure), and calculate each of the iterated integrals $\int_X \int_Y f(x, y)\, dy\, dx$ and $\int_Y \int_X f(x, y)\, dx\, dy$. Is the function $f(x, y)$ integrable on $[0, 1] \times [0, 1]$? Justify fully.

12. Let $f \in L_1(\mathbb{R})$. Show that $g(x) = \int_{-\infty}^{\infty} \sin(xt) f(t) dt$ is continuous and bounded.

13. Let T be a surjective linear map from a Banach space X onto a Banach space Y such that
$$\|Tx\| \ge \frac{1}{2017} \|x\|$$
for all $x \in X$. Show that T is bounded.

14. Find a linear functional L on $C[-2, 2]$ such that there does not exist a function $g \in C[-2, 2]$ with $\|g\|_\infty = 1$ and $|L(g)| = \|L\|$.