QUALIFYING EXAMINATION II
IN
MATHEMATICAL STATISTICS

SATURDAY, MAY 14, 2016

Examiners: Drs. K. M. Ramachandran and G. S. Ladde

INSTRUCTIONS: a. The **quality** is more important than the **quantity**.
b. Attempt **at least 2 problems from each part totaling at most 5 problems**.
c. In the **absence of any Tables**, students are expected to provide the solution of the data oriented problems in the form of the **problem solving process**.
d. Students are also expected to exhibit the **reading, writing and problem solving abilities**.
e. Just the **mechanical work with an answer** (if any) is **not enough to receive the full credit**.

PART-1

1. Which of the following statements are TRUE or FALSE? Justify your responses.
   (a) A hypothesis is a statement about a population parameter.
   (b) The hypothesis problem decomposes the population parameter set into two subsets.
   (c) The hypothesis problem decomposes the sample space into two subsets.
   (d) An interval estimate is less precision with a greater measure of confidence.

2. Let $X_1, X_2, ..., X_i, ..., X_n$ be a random sample from exponential (β) population. Find:
   a). the joint pdf of the sample,
   b). $E[\overline{X}]$,
   c). $Var(\overline{X})$,
   d). $E[S^2]$.
   e). Based on your response in b)-e), what conclusions can you draw?

3. Which of the following statements are TRUE or FALSE? Justify your responses.
   (a) The true value is guaranteed by the point estimator rather than interval estimator.
   (b) For level α test, $P(\text{Reject H}_0 |\ \theta \in \Theta_o) \leq \alpha$ and $P(\text{Accept H}_0 |\ \theta \in \Theta_o) \geq 1 - \alpha$ provides a basis for to determine the α level region parameter estimation.
   (c) For $a < b$, the coverage probability of $[aY,\ bY]$ dependents on parameter θ.
   (d) For $a < b$, the coverage probability of $[a + Y,\ b + Y]$ is independent of θ.

4. Let $\mathrm{H_o}$, $\mathrm{H_1}$, Θ and Θ_o are defined in a hypothesis testing problem. Suppose that $\mathcal{R}$ stands for the rejection region, and $\beta(\theta) = P_\theta(X \in \mathcal{R})$.
   (a) Show that:
      (i) $P_\theta(X \in \mathcal{R})$ is **Type I Error**, and (ii) $P_\theta(X \in \mathcal{R}) = 1$ minus **Type II Error**.
   (b) For $0 \leq \alpha \leq 1$, find a Level α Test.
   (c) Let $X \sim$ binomial $(2, p)$, and let $\mathrm{H_o}$: $p \in \{p: p \leq \frac{1}{2}\}$ versus $\mathrm{H_1}$: $p \in \{p: p \geq \frac{1}{2}\}$ and reject the $\mathrm{H_o}$ if $X = 2$. Find the:
      (i) power function for this test, (ii) Type I Error, and (iii) Type II Error

5. A random variable $Q(X, \theta) = Q(X_1, X_2, .., X_n, \theta)$ is a pivotal quantity (pivot), if the distribution of $Q(X, \theta)$ is independent of all parameters, that is, if $X \sim F(x|\theta)$, then $Q(X, \theta)$ has the same distribution for all values of θ.
   (a) For given α belonging to the set $(0, 1)$, determine a confidence region that is independent of θ.
   (b) For given $X_1, X_2, .., X_n$ are iid exponential(λ) with sufficient statistics $T(X) = \sum_{i=1}^{n} X_i$ for λ and $T \sim$ gamma(n, λ).
   Show that : (i) $Q(X, \lambda) = \frac{2}{\lambda} T(X)$ and (ii) $Q(X, \lambda) \sim \chi^2_{2n}$.
   (c) For given α belonging to the set $(0, 1)$, determine a confidence interval for $\lambda = 1$

PART-2

1. Let X be normal random variable with mean μ and variance σ^2. Let us further assume that $\mu \sim N(\mu_p, \sigma_p^2)$ and its prior distribution is $\pi(\mu)$. μ_p and σ_p^2 are assumed to be either known or estimated.

   a) Find the marginal distribution of the given random.
   b) Define the likelihood function, $L(\theta|x)$, where x is a realization of random variable X.
   c) Find a posterior distribution of θ, $\pi(\theta|x)$.
   d) Find the α-level credible interval μ.

2. Let $f(x|\theta)$ be the density of X.
   a. Determine the Fisher information $I(\theta)$, (i) $\theta \in R^m$,for $m \geq 1$.
   b. Find the Jeffrey's prior.
   c. Is the prior in (c) proper or improper. Justify.

3. Prove or disprove the following statement:
   a. Some exponential distribution has conjugate prior distribution.
   b. The likelihood function plays insignificant role in the conjugate prior family.

4. Let $Y \sim f_Y$ and $V \sim f_V$. Further assume f_Y and f_V have common support with $M = \sup \frac{f_Y(y)}{f_V(y)} < \infty$. To generate a random variable $Y \sim f_Y$: one needs to apply the following three steps: a_1) Generate $U \sim (0,1)$ and $V \sim f_V$, independently; a_2) If $U < \frac{f_Y(V)}{M f_V(V)}$, set $Y = V$ and stop; a_3) Otherwise repeat step (a_1) until the process stops. Show that:
(a) the generated random variable Y has a desired cdf;
(b) M is expected number of trials.
(c) What can you say about distributions V's of two consecutive trials?

5. Let X be a random variable with exponential(λ) distribution.
a. Find:
(i) $F_X(x)$,
(ii) $F_X^{-1}(x)$ (if it exists).
b. Is it possible simulate the exponential(λ)? Justify.
c. If the answer to question in b is "YES", then is it possible to simulate gamma (α, β)? Justify.

GOOD LUCK

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFYING EXAMINATION II
IN
MATHEMATICAL STATISTICS

SATURDAY, SEPTEMBER 24, 2016

Examiners: Drs. K. M. Ramachandran and G. S. Ladde

INSTRUCTIONS:
a. The **quality** is more important than the **quantity**.
b. Attempt **at least 2 problems from each part totaling at most 5 problems**.
c. In the **absence of any Tables**, students are expected to provide the solution of the data oriented problems in the form of the **problem solving process**.
d. Students are also expected to exhibit the **reading, writing and problem solving abilities**.
e. Just the **mechanical work with an answer** (if any) is **not enough to receive the full credit**.

PART-1

1. Let X be a real-valued random variable defined on a complete probability space $(\Omega, \mathfrak{F}, P)$ with singleton set range (R($X) = \{x\}$ any $x \in$ R; where R is a set of real numbers).

   a. Find: a distribution function F_X of X. Justify
   b. Find a probability density function f_X of X (if exists).
   c. Draw the sketches of F_X and f_X with justifications.
   d. What conclusion can be drawn from Parts (a), (b) and (c)? Justify.

2. Let $X_1, X_2, .., X_n$ be a Bernoulli-type random sample drawn from a population mean $\mu \in [0,1]$ and variance $\sigma^2 \in [0,1]$. $\overline{X}$ and S^2 sample mean and variance of the random sample.

   a. Find the joint distribution of the random sample ;
   b. Find: $E[\overline{X}]$;
   c. Find: Var $[\overline{X}]$.
   d. Show that $E[S^2] = \sigma^2$,
   e. Based on your response in (a)-(d), what conclusions can you draw.

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3. Let $X_1, X_2, ..., X_n, ...$ be a sequence of iid random variables with mean ($E[X_n] = \mu$, $Var(X_n) = \sigma^2$). Show that:

(a) S_n^2 converges in probability to σ^2 as $n \to \infty$, whenever $Var(S_n^2) \to 0$.
(b) Prove or disprove convergence or divergence of S_n.
(c) On the basis of (a) and (b),what types of conclusion can you draw about S^2 and S_n?

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4. Let $X_1,\ X_2,\ ..,\ X_n$ be a random sample defined in Problem 2. Let $T(X)$ be defined by: $T(X) = \sum_{i=1}^{n} X_i$. Prove or disprove the following statements:
   a. The distribution of $T(X)$ is an exponential family.
   b. $T(X)$ is sufficient statistics.
   c. The Bernoulli MLE of μ is $\mu\hat{} = \frac{\sum_{i=1}^{n} x_i}{n}$.

5. Let $X_1,\ X_2,\ ..,\ X_n$ be a random sample, $X = (X_1,\ X_2,\ ..,\ X_n)^T, X = x \in \chi \subseteq R^n$ be observed sample set. Let $f(x|\theta)$ be the joint pdf of the given the sample X, and let $\pi(\theta)$ be a prior distribution of θ. Let Θ be an entire parameter set in R and $\Theta_0 \neq \phi$. Moreover, let $P(\theta \in \Theta_0|x) = P($ H$_0$ is true$|x)$ and $P(\theta \in \Theta_0^c|x) = P($ H$_1$ is true$|x)$.
   (a) Define the "action set" with respect to the "Bayesian hypothesis testing problem".
   (b) Is the "hypothesis test" identical with a "decision rule"? Justify.
   (c) Define the loss function in the "hypothesis testing problem".
   (d) Define the rejection region for the "Bayesian hypothesis testing problem".
   (e) Determine the costs for **Type I and II Error**

PART-2

1. Let X be a binomial random variable with mean $\mu \in [0, 1]$ and variance $\sigma^2 \in [0, 1]$, and let $X_1,\ X_2,\ ..,\ X_n$ be a corresponding random sample drawn from this populatiom. In the Bayesian analysis, it is assumed that the parameter μ is random variable. Let $p(y, \mu)$ be the joint distribution of (X, μ). The binomial sampling model is denoted by $p(y|\mu)$.

   a. Justify the exchangeability property of $p(y, \mu)$.
   b. Find the expression for the joint probability of X and.
   c. Find the marginal distribution of the given random.
   d. Define the likelihood function, $L(\mu|y)$, where x is a realization of random variable X.

2. Let $Y \sim$ Beta(α, β) and $V \sim f_V$. Further assume Beta(α, β) and f_V have common

$[0, 1]$ with $\alpha = 2.7, \beta = 6.3$ and $c = 2.67 = \sup f_Y(y) < \infty$. Let (U, V) be independent and uniformly distributed random variables to generate a random variable. Show that:

(a) $P\Big(\{\omega : V(\omega) \leq y \text{ and } U(\omega) \leq \frac{\text{Beta}(\alpha,\beta)(V(\omega))}{c}\}\Big) = \int_0^y \int_0^{\frac{\text{Beta}(\alpha,\beta)(V(\omega))}{c}} du dv.$

(b) $P\Big(\{\omega : U(\omega) \leq \frac{\text{Beta}(\alpha,\beta)(V(\omega))}{c}\}\Big) = \frac{1}{c}.$

(c) $P\Big(\{\omega : Y(\omega) \leq y\}\Big) = P\Big(\{\omega : V(\omega) \leq y\} \Big| \{\omega : U(\omega) \leq \frac{\text{Beta}(\alpha,\beta)(V(\omega))}{c}\}\Big).$

(d) What conclusions can you draw from the results (a)-(c)?

3. Let X be a binomial random variable corresponding to the random sample in Problem 1 (Part 2). In the Bayesian analysis, it is assumed that the parameter μ is random variable. Let $p(y, \mu)$ be the joint distribution of (X, μ). The binomial sampling model is denoted by $p(y|\mu)$. In addition assume that $p(\mu)$ is prior distribution μ.

a. Find a posterior distribution of μ, $p(\mu|y)$.
b. If $\mu \sim$ Beta (α, β), the find $p(\mu|y)$.
c. What conclusions can you draw from the conclusion of (b)?
d. Using (b) find $E[\mu|y]$ and Var$(\mu|y)$.
e. Show that $E[\mu|y] \approx$ Var$(\mu|y) \approx \frac{1}{n}\frac{y}{n}(1 - \frac{y}{n})$the α-level credible interval for μ.

4. Prove or disprove the following statement.

a. The normal distribution has conjugate prior distribution.
b. The Binomial distribution (n, μ) has no conjugate prior distribution.
c. The Poisson distribution has conjugate prior distribution.

5. Assume that all the conditions in Problem 1 (Part 2) are valid.

a. Is it possible to determine the Fisher information $I(\mu)$, $\mu \in R$? Justify.
b. If the answer to the question in (a) is "YES", then find the Jeffrey's prior density .
c. Based on your work in (b), is the prior in (b) proper or improper? Justify.

GOOD LUCK

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFYING EXAMINATION II
IN
MATHEMATICAL STATISTICS

SATURDAY, JANUARY 28, 2017

Examiners: Drs. K. M. Ramachandran and G. S. Ladde

INSTRUCTIONS: a. The **quality** is more important than the **quantity**.
b. Attempt **at least 2 problems from each part totaling at most 5 problems.**
c. In the **absence of any Tables**, students are expected to provide the solution of the data oriented problems in the form of the **problem solving process.**
d. Students are also expected to exhibit the **reading, writing and problem solving abilities**.
e. Just the **mechanical work with an answer** (if any) is **not enough to receive the full credit**.

PART-1

1. Prove or disprove the following statements:
   a. The distribution of statistics is population distribution.
   b. If the marginal densities of random variables are identical,then they are independent.
   c. $Var(a_1X_1 + a_2X_2) = a_1^2Var(X_1) + a_2^2Var(X_2) + 2a_1a_2Cov(X_1, X_2)$.

2. Let X_1, X_2, X_3 be random sample drawn from exponential(β) population. Find:

   (i) $X_{(1)}$
   (ii) $X_{(3)}$
   (iii) the sample range.

3. Let $X_1,\ X_2,\ ..,\ X_n$ be a sample, $X = (X_1,\ X_2,\ ..,\ X_n)^T, T(X)$ be a statistics, $X = x \in \chi \subseteq R^n$ be observed sample set, and $R(T)$ be the range of statistics T. Let $f(x|\theta)$ be the joint pdf of the given the sample, X

   (a) If $X_1,\ X_2,\ ..,\ X_n$ are Bernoulli random variables with parameter $0 < \theta < 1$ and $T(X) = \sum_{i=1}^{n} X_i$, then show that $T(X)$ is sufficient statistics.
   (b) If $X_1,\ X_2,\ ..,\ X_n$ are iid observations from discrete uniform distribution on $\{1,\ 2,\ 3,\ ...,\theta\}, \theta \in N$ and $T(X) = \max_i\{X_1,\ X_2,\ ..,\ X_n\}$, then show that $T(X)$ is sufficient statistics.

4. Let C be a class of tests for testing H_o: $\theta \in \Theta_o$ versus H_1: $\theta \in \Theta_o^c$. A test in class C, with power function $\beta(\theta)$, is a **Uniformly Most Powerful (UMP) Class C Test** if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Theta_o^c$ and every $\beta'(\theta)$ that is a powerful function of a test in class C. Let $\theta_o \in \Theta_o$, $\theta^c \in \Theta_o^c$ and and let $f(x|\theta_o)$and $f(x|\theta^c)$ corresponding pdf/pmf, respectively. Further assume that for $k \geq 0$:

(4.1) $x \in \mathcal{R}$ if $f(x|\theta^c) > kf(x|\theta_o)$ and $x \in \mathcal{R}$ if $f(x|\theta^c) < kf(x|\theta_o)$,

(4.2) $\alpha = P_\theta(X \in \mathcal{R})$ for given α belonging to the set $(0, 1)$.

(a) Show that:

(i) (4.1) and (4.2) are sufficient conditions for a UMP level α test;

(ii) (4.1) and (4.2) are necessary conditions for a UMP level α test.

(b) Give sufficient the conditions to determine a UMP Binomial Test.

PART-2

1. Under the Bayesian approach, for given $X = x$, the $\pi(\theta|x)$ is posterior distribution of θ. For $A \subseteq \Theta$, the credible probability of A is defined by:

$P(\theta \in A|x) = \int_A \pi((\theta|x)d\theta$.

(a) Is there any relationship between the credible set and confidence set? Justify.

(b) For given $X_1, X_2, .., X_n$ be random sample with Poisson (λ) and λ has a gamma (a, b) prior. The posterior pdf of λ is gamma $(a + T(x), [n + \frac{1}{b}]^{-1})$, where $T(X) = \sum_{i=1}^{n} X_i$. Find the credible set for λ.

(c) In addition to the conditions in (b), assume that: $n = 10$ and $T(x) = 6$. Find 90% credible set for λ.

2. Let X be normal random variable with mean μ and variance σ^2. Let us further assume that $\mu \sim N(\mu_p, \sigma_p^2)$ and its prior distribution is $\pi(\mu)$. μ_p and σ_p^2 are assumed to be either known or estimated.

(a) Find the marginal distribution of the given random.

(b) Define the likelihood function, $L(\theta|x)$, where x is a realization of random variable X.

(c) Find a posterior distribution of θ, $\pi(\theta|x)$.

(d) Find the α-level credible interval μ.

3. Assume that all the given statements in Problem # 2 remain unchanged. Further assume that $\mu_p = 100,\ \sigma_p = 15,\ \sigma = 10$ and $x = 115$.

   (a) Find $E[\mu|x]$ and $E[(\mu - E[\mu])^2|x]$.
   (b) Find the precision of the normal distribution.
   (c) Show that the $E[\mu] = E[E[\mu|x]]$ and interpret it.
   (d) Show that $E[\text{var}(\mu|x)] = \text{var}(\mu) - \text{var}(E[\mu|x])$ and interpret it

4. Let $X_1, X_2, ..., X_n$ be a random sample of size n from a population $N(\mu,\ \sigma^2)$ with $\sigma^2 = 4$. Further assume that $\mu \sim N(0,\ 1)$.

   (a) Find $E[\mu|x]$ and $E[(\mu - E[\mu|x])^2|x]$.
   (b) Find the precision of the normal distribution.
   (c) Find 95% credible interval for μ

GOOD LUCK

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFYING EXAMINATION II
IN
MATHEMATICAL STATISTICS

SATURDAY, MAY 13, 2017

Examiners: Dr's. K. M. Ramachandran and G. S. Ladde

INSTRUCTIONS:
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c. In the **absence of any Tables**, students are expected to provide the solution of the data oriented problems in the form of the **problem solving process**.
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1. Let X be a real-valued random variable defined on a complete probability space $(\Omega, \mathfrak{F}, P)$ with singleton set range (R($X) = \{x\}$ any $x \in$ R; where R is a set of real numbers).

   a. Find: a distribution function F_X of X. Justify
   b. Find a probability density function f_X of X (if exists).
   c. Draw the sketches of F_X and f_X with justifications.
   d. What conclusion can be drawn from Parts (a), (b) and (c)? Justify.

2. Let $X_1, X_2, .., X_n$ be a sample, $X = (X_1, X_2, .., X_n)^T$ from iid binomial (n, p). It is that both n and p are unknown .
   (a) Use the Method of Moment to find point estimators for both parameters n and p.
   (b) Identify the limitations and drawbacks of this method

3. Let T be any unbiased estimator of $\tau(\theta)$, and let W be a sufficient statistics for θ. Define $\phi(W) = E_\theta[T|W]$. Show that:
   (a) $\phi(W)$ is unbiased estimator of $\tau(\theta)$,
   (b) $\text{Var}_\theta(T) = \text{Var}_\theta(\phi(W)) + E_\theta[\text{Var}_\theta(T|W)]$,
   (c) $\text{Var}_\theta(\phi(W)) \leq \text{Var}_\theta(T)$,
   (d) Is the inequality in Part 3(c) uniformly in θ? Justify.
   (e) Is the $\phi(W)$ uniformly better unbiased estimator of $\tau(\theta)$? Justify.

4. Let H_o, H_1, Θ and Θ_o are defined in a hypothesis testing problem. Suppose that $\mathcal{R}$ stands for the rejection region, and $\beta(\theta) = P_\theta(X \in \mathcal{R})$.
   (a) Show that:
      (i) $P_\theta(X \in \mathcal{R})$ is **Type I Error**, and (ii) $P_\theta(X \in \mathcal{R}) = 1$ minus **Type II Error**.
   (b) For $0 \le \alpha \le 1$, find a α Level α Test.
   (c) Let $X \sim$ binomial $(2, p)$, and let H_o: $p \in \{p$: $p \le \frac{1}{2}\}$ versus H_1: $p \in \{p$: $p \ge \frac{1}{2}\}$ and reject the H_o if $X = 2$. Find the:
      (i) power function for this test, (ii) Type I Error, and (iii) Type II Error.

PART-2

1. Let X and Y be two random variables. It is known that: $\text{Var}(X) = E[(X - E[X])^2]$ $= E[X^2] - (E[X])^2]$, $E[X] = E[E[X|Y]]$ and $\text{Var}(X|Y) = E[(X - E[X|Y])^2|Y]$.
   Show that:
   (a) $\text{Var}(X|Y) = E[X^2|Y] - (E[X|Y])^2$,
   (b)$\text{Var}(E[X|Y]) = \text{E}[(E[X|Y])^2] - (E[E[X|Y]])^2$,
   (c) $\text{Var}(X) = E[E[(X - E[X])^2|Y]] = E[E[X^2|Y] - (E[E[X|Y]]\,)^2$,
   (d) $\text{Var}(X) = \text{Var}(E[X|Y]) + E[\text{Var}(X|Y)]$.

2. Let X be a binomial random variable corresponding to the random sample in Problem 2 (Part 1). In the Bayesian analysis, it is assumed that the parameter μ is random variable. Let $p(y, \mu)$ be the joint distribution of (X, μ). The binomial sampling
   model is denoted by $p(y|\mu)$.

   a. Justify the exchangeability property of $p(y, \mu)$.
   b. Find the expression for the joint probability of X and.
   c. Find the marginal distribution of the given random.
   d. Define the likelihood function, $L(\mu|y)$, where x is a realization of random variable X.

3. Assume that all the conditions in Problem 2 are valid. In addition assume that $p(\mu)$ is prior distribution μ.

   a. Find a posterior distribution of μ, $p(\mu|y)$.
   b. If $\mu \sim$ Beta (α, β), the find $p(\mu|y)$.
   c. What conclusions can you draw from the conclusion of (b)?
   d. Using (b) find $E[\mu|y]$ and $\text{Var}(\mu|y)$.
   e. Show that $E[\mu|y] \approx \text{Var}(\mu|y) \approx \frac{1}{n}\frac{y}{n}(1 - \frac{y}{n})$the α-level credible interval for μ.

4. Let $Y \sim$ Beta(α, β) and $V \sim f_V$. Further assume Beta(α, β) and f_V have common

$[0,1]$ with $\alpha = 2.7, \beta = 6.3$ and $c = 2.67 = \sup f_Y(y) < \infty$. Let (U, V) be independent and uniformly distributed random variables to generate a random variable. Show that:

(a) $P\Big(\{\omega : V(\omega) \leq y \text{ and } U(\omega) \leq \frac{\text{Beta}(\alpha,\beta)(V(\omega))}{c}\}\Big) = \int_0^y \int_0^{\frac{\text{Beta}(\alpha,\beta)(V(\omega))}{c}} du dv.$

(b) $P\Big(\{\omega : U(\omega) \leq \frac{\text{Beta}(\alpha,\beta)(V(\omega))}{c}\}\Big) = \frac{1}{c}$.

(c) $P\Big(\{\omega : Y(\omega) \leq y\}\Big) = P\Big(\{\omega : V(\omega) \leq y\} \Big| \{\omega : U(\omega) \leq \frac{\text{Beta}(\alpha,\beta)(V(\omega))}{c}\}\Big).$

(d) What conclusions can you draw from the results (a)-(c)?

GOOD LUCK

DEPARTMENT OF MATHEMATICS AND STATISTICS
QUALIFYING EXAMINATION II
IN
MATHEMATICAL STATISTICS

SATURDAY, SEPTEMBER 30, 2017

Examiners: Dr's. K. M. Ramachandran and G. S. Ladde

INSTRUCTIONS: a. The **quality** is more important than the **quantity**.
b. Attempt **at least 2 problems from each part totaling at most 5 problems**.
c. In the **absence of any Tables**, students are expected to provide the solution of the data oriented problems in the form of the **problem solving process**.
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PART-1

1. Let $\{X_n\}_1^\infty$ be a sequence of random variables defined on the complete probability space $(\Omega, \mathfrak{F}, P)$ with the range of each term X_n is singleton set $\{1 + \frac{1}{n}\}$. Find:

(a) $\{F_{X_n}\}_1^\infty$ and $\{f_{X_n}\}_1^\infty$ (if possible). Justify.
(b) Does$\{X_n\}_1^\infty$ converge in distribution? Justify.
(c) Does$\{X_n\}_1^\infty$ converge in almost sure sense? Justify.
(d) Based on your responses to above questions, can you draw any conclusions?

2. Let $X_1, X_2, .., X_n$ be a random sample from Binomial (n, p) distribution, where both n and p are unknown.

(a) By employing the Method of Moment, find estimators for both n and p.
(b) Discuss a situation in which both n and p are unknown.
(c) Five realizations of a binomial (n, p) experiment are observed.
(i) The First data set is: 16, 18, 22, 25, 27, and
(ii) the second data set is: 16, 18, 22, 25, 28.
For these data sets, using the method of moment,calculate the estimates for n and p.
(d) Compare the estimates in Problem 2(c).
(e) What kind of conclusions can you draw from Problem 2(d)?

3. Let g be twice continuously differentiable function defined on R into itself. Let X be a random variable defined on the complete probability space $(\Omega,\ \mathfrak{F},\ P)$ with mean μ. Show that:

(a) $E[g(X)] \approx g(\mu)$. Justify.
(b) $\text{Var}(g(X)) \approx \ |\frac{d}{dx}g(\mu)|^2\text{Var}(X)$. Justify.

4. Let $X_1,\ X_2,\ ..,\ X_n$ be a random sample, $X = (X_1,\ X_2,\ ..,\ X_n)^T, X = x \in \chi \subseteq R^n$ be observed sample set. Let $f(x|\theta)$ be the joint pdf of the given the sample, X, and let Θ be an entire parameter set in R. For $\Theta_o \neq \phi$, let us define $\lambda(x) = \frac{\sup_{\Theta_o} L(\theta|x)}{\sup_{\Theta} L(\theta|x)}$.

(a) Show that:

(i) $0 \leq \lambda(x) \leq 1$, for all $x \in \chi$, (ii) As $\sup_{\Theta} L(\theta|x)$ increases, $\lambda(x)$ decreases, (iii) $\lambda(x) = \frac{L(\hat{}\theta_o|x)}{L(\hat{}\theta|x)}$, (iv) $\lambda(x) = \frac{\sup_{\Theta_o} g(T(x)|\theta)}{\sup_{\Theta} g(T(x)|\theta)} = \lambda^*(T(x))$, where $T(X)$ is sufficient statistics and $f(x|\theta) = g(T(x)|\theta)h(x)$;

(b) Under what condition(s) on Θ_o, $\lambda(X)$ is called as the **Likelihood Ratio Test Statistics**?

(c) Assuming that $\lambda(X)$ is Likelihood Ratio Test statistics, what can you say about : $\{x : \lambda(x) \leq c\}$ for any number c satisfying $0 \leq c \leq 1$?

PART-2

1. Under the Bayesian approach, for given $X = x$, the $\pi(\theta|x)$ is posterior distribution of θ. For $A \subseteq \Theta$, the credible probability of A is defined by:

$P(\theta \in A|x) = \int_A \pi((\theta|x)d\theta$.

(a) Is there any relationship between the credible set and confidence set? Justify.
(b) For given $X_1,\ X_2,\ ..,\ X_n$ be random sample with Poisson (λ) and λ has a gamma (a, b) prior. The posterior pdf of λ is gamma $(a + T(x), [n + \frac{1}{b}]^{-1})$, where $T(X) = \sum_{i=1}^{n} X_i$. Find the credible set for λ.
(c) In addition to the conditions in (b), assume that: $n = 10$ and $T(x) = 6$. Find 90% credible set for λ.

2. Let $X_1, X_2, ..., X_n$ be a random sample of size n from a population $N(\mu,\ \sigma^2)$ with $\sigma^2 = 4$. Further assume that $\mu \sim N(0,\ 1)$.

(a) Find $E[\mu|x]$ and $E[(\mu - E[\mu|x])^2|x]$.
(b) Find the precision of the normal distribution.
(c) Find 95% credible interval for μ.

3. Let X be normal random variable with mean μ and variance σ^2. Let us further assume that $\mu \sim N(\mu_p,\ \sigma_p^2)$ and its prior distribution is $\pi(\mu)$. μ_p and σ_p^2 are assumed to be either known or estimated.

(a) Find the marginal distribution of the given random.
(b) Define the likelihood function, $L(\theta|x)$, where x is a realization of random variable X.
(c) Find a posterior distribution of θ, $\pi(\theta|x)$.
(d) Find the α-level credible interval μ.

4. Assume that all the given statements in Problem # 2 remain unchanged. Further assume that $\mu_p = 100,\ \sigma_p = 15,\ \sigma = 10$ and $x = 115.$

(a) Find $E[\mu|x]$ and $E[(\mu - E[\mu])^2|x]$.
(b) Find the precision of the normal distribution.
(c) Show that the $E[\mu] = E[E[\mu|x]]$ and interpret it.
(d) Show that $E[\text{var}(\mu|x)] = \text{var}(\mu) - \text{var}(E[\mu|x])$ and interpret it

GOOD LUCK