Department of Mathematics and Statistics
University of South Florida

TOPOLOGY QUALIFYING EXAM
May 14, 2016

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Let $K = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Let $\mathcal{B}_K = \{(a,b),\ (a,b) \setminus K \mid a, b \in \mathbb{R}, a < b\}$.
   Show that $\mathcal{B}_K$ forms a basis of a topology, $\mathbb{R}_K$, on $\mathbb{R}$, and that $\mathbb{R}_K$ is strictly finer than the standard topology of $\mathbb{R}$.

2. Recall that a space X is *countably compact* if any countable open covering of X contains a finite subcovering. Prove that X is countably compact if and only if every nested sequence $C_1 \supset C_2 \supset \cdots$ of closed nonempty subsets of X has a nonempty intersection.

3. Assume that all singletons are closed in X. Show that if X is regular, then every pair of points of X has neighborhoods whose closures are disjoint.

4. Let $X = \mathbb{R}^2 \setminus \{(0,n) \mid n \in \mathbb{Z}\}$ (integer points on the y-axis are removed) and $Y = \mathbb{R}^2 \setminus \{(0,y) \mid y \in \mathbb{R}\}$ (the y-axis is removed) be subspaces of $\mathbb{R}^2$ with the standard topology. Prove or disprove: There exists a continuous surjection $X \to Y$.

5. Let X and Y be metrizable spaces with metrics d_X and d_Y respectively. Let $f : X \to Y$ be a function. Prove that f is continuous if and only if given $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$.

6. For a surjective map $p : X \to Y$, recall that $Z \subset X$ is *saturated with respect to* p if Z contains every set $p^{-1}(\{y\})$ that it intersects, for any $y \in Y$. Assume that p is a quotient map and let Z be a saturated subspace of X with respect to p. Prove that if p is an open map then the map $q : Z \to f(Z)$, obtained by restricting p, is a quotient map.

7. Let $f : X \to Y$ be a continous function where Y is a Hausdorff space.
   Prove or disprove the following: The graph of f is a closed subspace of $X \times Y$.

Section II: ALGEBRAIC TOPOLOGY

1. Let S^1 be the complex numbers with unit norm. Let $f_n : S^1 \to \mathbb{C} \setminus \{0\}$ be defined by $f(z) = z^n$ for a positive integer n. Prove that f_n is not null-homotopic.

2. Let X be the union of the twelve edges, together with the eight vertices, of the unit cube in 3-space. Determine the fundamental group of X.

3. A space X is said to be contractible if the identity map $i_X : X \to X$ is nulhomotopic. Prove that that a contractible space is path connected.

4. Prove that the n-sphere S^n is simply connected for $n \geq 2$.

5. Let $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ be continuous maps such that $h = g \circ f$. Prove that if g and h are covering maps then f is also a covering map.

6. Let A, B and C be pairwise distinct points of the plane $\mathbb{R}^2$.
   Prove or disprove: $\mathbb{R}^2 \setminus \{A, B, C\}$ is a deformation retract of $\mathbb{R}^2 \setminus \{A, B\}$.

7. Let $p \in S^1$ and let X be the quotient space of the torus $\mathbb{T} = S^1 \times S^1$ by the subspace $S^1 \times \{p\}$ (where $S^1 \times \{p\}$ is identified to a point). Determine all homology groups of the space X.

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TOPOLOGY QUALIFYING EXAM
September 24, 2016

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Prove or disprove: Let A, B be subspaces of X, Y, respectively. Then the product topology on $A \times B$ is the same as the subspace topology.

2. Let $X = \mathbb{R}^2 \setminus \{0\}$ and $Y = S^1 \cup ([0,1] \times \{0\}) \subset \mathbb{R}^2$ be the union of the unit circle and the unit interval on the x-axis. Prove or disprove: X is homeomorphic to Y.

3. Outline a proof that any second countable regular space is metrizable.

4. Let A, B, and A_α be subsets of a topological space. Prove or disprove each of the following three equalities:

   (a) $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$.

   (b) $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$.

   (c) $\overline{\cap A_\alpha} = \cap \overline{A_\alpha}$.

5. Let $f : X \to Y$ be a surjective continuous map. Let $X^* = \{f^{-1}(\{y\}), \quad y \in Y\}$ be given the quotient topology. Let $g : X^* \to Y$ be the bijective continuous map induced from f (that is, $f = g \circ p$, where $p : X \to X^*$ is the projection map).

   Prove that the map $g : X^* \to Y$ is a homeomorphism if and only if f is a quotient map.

6. Show that any compact Hausdorff space is normal.

7. Let $f : S^1 \to \mathbb{R}$ be a continuous function from the unit circle to the real line.

   (1) Prove that there exists $z_0 \in S^1$, such that $f(z_0) = f(-z_0)$.

   (2) Determine whether f can be surjective.

Section II: ALGEBRAIC TOPOLOGY

1. Let L_i, $i = 1, 2, 3$, be three paiwise disjount lines in $\mathbb{R}^3$. Determine the fundamental group of $\mathbb{R}^3 \setminus (L_1 \cup L_2 \cup L_3)$.

2. State the equivalence of covering spaces, and classify covering spaces of the circle S^1.

3. Exhibit, with a proof, a space X such that the fundamental group $\pi_1(X)$ and the first homology group $H_1(X)$ are not isomorphic.

4. State the definition of *homotpy type* of a space. Then prove that the space $\mathbb{R}^{m+1} \setminus \{0\}$ and the sphere S^m have the same homotopy type.

5. For $m \geq 3$, prove that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^2$.

6. For a given non-negative integer n, give the list (with an explanation) of all compact connected surfaces X (with or without boundary) such that the first homology group $H_1(X)$ has rank n.

7. Determine the homology groups of the following chain complex, where $C_n = \mathbb{Z}$, ∂_{2n} are multiplication by 2 and ∂_{2n+1} are zero maps for each $n \geq 0$:

$$\cdots \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3=0} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1=0} \mathbb{Z} \longrightarrow 0.$$

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TOPOLOGY QUALIFYING EXAM
January 28, 2017

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Let X be a space satisfying the T_1-axiom, and let A be a subset of X. Prove that $x \in X$ is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

2. Let (X_1, d_1) and (X_2, d_2) be metric spaces with metric topologies. Let $x = (x_1, x_2), y = (y_1, y_2) \in X_1 \times X_2$. Show that $d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$ defines a metric on $X_1 \times X_2$, that induces the product topology.

3. Let A and B be proper subsets of connected spaces X and Y, respectively. Prove that $(X \times Y) \setminus (A \times B)$ is connected.

4. Let Y_i, $i = 1, 2$, be compact Hausdorff spaces with a subspace X such that $Y_i \setminus X$ consists of a single point. Prove that there is a homeomorphism $h : Y_1 \to Y_2$ such that $h|_X$ is the identity map on X.

5. Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. State the definition of uniform convergence for (f_n), and prove that if (f_n) converges uniformly to f, then f is continuous.

6. Let $f : X \to Y$ be a quotient map. Suppose that Y is connected and for every $y \in Y$, $f^{-1}(y)$ is connected. Prove that X is connected.

7. Prove or disprove: every metric space is normal.

Section II: ALGEBRAIC TOPOLOGY

1. Determine the fundamental group of the wedge (one-point union) of two tori, $T \vee T$.

2. Let n be a positive integer. Find all closed (i.e., compact without boundary) surfaces F such that there exists a surjective homomorphism $H_1(F) \to (\mathbb{Z}_2)^n$.

3. Construct a chain complex $\mathcal{C} = (C_n, \partial_n)$ such that $H_0(\mathcal{C}) \cong \mathbb{Z}$, $H_i(\mathcal{C}) \cong \mathbb{Z}_2$ for $i = 1, 2$ and $H_j(\mathcal{C}) = 0$ for $j \neq 0, 1, 2$.

4. Prove or disprove: There exists a continuous map $P \times \cdots \times P \to S^1$ that is not null-homotopic, where P denotes the projective plane, n is a positive integer, $P \times \cdots \times P$ is the n-fold product and S^1 denotes the circle.

5. Let $p : (E, e_0) \to (B, b_0)$ be a covering map. Show that any path in B beginning at b_0 has a unique lifting to a path in E beginning at e_0.

6. State the definition of homotopy equivalence of continuous functions. Then prove that a homotopy equivalence induces an isomorphism on the fundamental groups.

7. Let $p : (E, e_0) \to (B, b_0)$ be a covering map and assume that E is simply connected. Prove that the group of covering transformations of (E, p, B) is isomorphic to the fundamental group of B.

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TOPOLOGY QUALIFYING EXAM
May 13, 2017

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Let (X, d) be a metric space. Let $B = \overline{B}_d(x, r) = \{y \in X \mid d(x, y) \leq r\}$ be a closed ball, where $x \in X$, $r > 0$.

   Prove or disprove: For any subset A of X, $\mathrm{C}\ell_B(A \cap B)$ is compact, where $\mathrm{C}\ell_B(C)$ denotes the closure of C in B.

2. Recall that a space is Lindelöf if every open covering contains a countable subcover.

   (a) Prove that a space is Lindelöf if and only if every covering by basis elements contains a countable subcover.

   (b) Prove that the second countability implies Lindelöf.

3. Prove or disprove: If there exists a continuous bijection between two topological spaces, then they are homeomorphic.

4. Show that if Y is compact, then the first projection $p_1 : X \times Y \to X$ is a closed map.

5. Give an example, with proofs, of connected space that is not path-connected.

6. Let $f : X \to Y$ be a continuous map, where Y is an ordered set with the order topology. Prove that if X is compact, then there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

7. Prove that every regular space with a countable basis is normal.

Section II: ALGEBRAIC TOPOLOGY

1. Determine, with a proof, the fundamental group of the wedge (one-point union) of two projective planes, $\mathbb{P}^2 \vee \mathbb{P}^2$.

2. Let K be the Klein bottle, and $X = K\#K\#K$. Find all pairs of closed surfaces F_1 and F_2 such that $F_1\#F_2$ is homeomorphic to X. Specify F_i, $i = 1, 2$, by orientability and genus, and provide a proof.

3. Let $\{C_n, \partial_n\}$, $n \in \mathbb{Z}_{\geq 0}$, be a chain complex such that $C_1 = \mathbb{Z}$, $C_2 = \mathbb{Z} \oplus \mathbb{Z}$ and $C_n = 0$ for $n \neq 1, 2$, where $\partial_2(m, n) = 2m + 2n$ for $(m, n) \in C_2$. Determine all homology groups.

4. Prove or disprove: There exists a retraction of a torus T onto a subspace of T homeomorphic to the bouquet of three circles.

5. A space is called *contractible* if the identity map $i : X \to X$ is nulhomotopic. Prove that a retract of a contractible space is contractible.

6. Show that if $n > 1$, then any continuous map $f : S^n \to T$ from the n-sphere to the torus is nullhomotopic.

7. Assume the theorem that if $h : S^1 \to S^1$ is a continuous antipode-preserving map ($h(-x) = -h(x)$ for all $x \in S^1$), then h is not nullhomotopic.

   (a) Show that there is no continuous antipode-preserving map $g : S^2 \to S^1$.

   (b) Show that for any continuous map $f : S^2 \to \mathbb{R}^2$, there exists $x \in S^2$ such that $f(x) = f(-x)$.

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TOPOLOGY QUALIFYING EXAM
September 23, 2017

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Prove that if A and B are disjoint compact subspaces of the Hausdorff space X, then there exists disjoint open sets U and V containing A and B, respectively.

2. Recall that a space X is called *completely regular* if one-point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

   Prove that a product (of any cardinality) of completely regular spaces is completely regular.

3. Prove that the components of a space X are connected disjoint subspaces whose union is X, such that each nonempty connected subspace of X intersects only one of them.

4. Let X be a topological space and $A \subset X$. Prove that if there is a sequence of points of A converging to x, then $x \in \overline{A}$. Prove also that the converse holds if X is first countable.

5. Prove that a space X is Hausdorff *if and only if* the diagonal $\{(x, x) \mid x \in X\}$ is closed in $X \times X$.

6. Let $\mathbb{R}^\omega$ be the Euclidean space of countably infinite dimension. Consider $\mathbb{R}^\omega$ with the box topology. Prove or disprove that the function f from $\mathbb{R}$ to $\mathbb{R}^\omega$ is continuous, where

$$f(x) = (x, \frac{x}{2}, \frac{x}{3}, \ldots).$$

7. Prove that $\mathbb{R}^\omega$ with product topology is connected.

Section II: ALGEBRAIC TOPOLOGY

1. Let $T = S^1 \times S^1$ be a torus and $x \in T$. Prove or disprove: There exists a continuous surjective map $f : T \to T$ such that the induced homomorphism $f_* : H_1(T, x) \to H_1(T, x)$ is the zero-map (meaning that the image is $\{0\}$).
2. Prove that the free group of rank 2 contains the free group of rank n as a subgroup for any positive integer $n > 2$.
3. Let X be the union of two copies of the 3-sphere S^3 having a single point in common. Compute the fundamental group of the space X.
4. Prove that if $f : S^1 \to S^1$ is nullhomotopic then f has a fixed point and f maps some x to its antipode $-x$.
5. Find a covering space of the torus $S^1 \times S^1$ corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by the element $m \times 0$, where m is a positive integer.
6. Determine the homology groups of the Klein bottle.
7. Let T be the torus and K be the Klein bottle. Find all pairs (S_1, S_2) of closed (i.e., compact without boundary) surfaces S_1 and S_2 such that $T\#S_1$ is homeomorphic to $K\#S_2$ and the genus of the resulting surface is less than 7.