Department of Mathematics and Statistics
University of South Florida

TOPOLOGY QUALIFYING EXAM
May 14, 2016

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Let $K = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Let $\mathcal{B}_K = \{(a, b),\ (a, b) \setminus K \mid a, b \in \mathbb{R}, a < b\}$.
   Show that $\mathcal{B}_K$ forms a basis of a topology, $\mathbb{R}_K$, on $\mathbb{R}$, and that $\mathbb{R}_K$ is strictly finer than the standard topology of $\mathbb{R}$.

2. Recall that a space X is *countably compact* if any countable open covering of X contains a finite subcovering. Prove that X is countably compact if and only if every nested sequence $C_1 \supset C_2 \supset \cdots$ of closed nonempty subsets of X has a nonempty intersection.

3. Assume that all singletons are closed in X. Show that if X is regular, then every pair of points of X has neighborhoods whose closures are disjoint.

4. Let $X = \mathbb{R}^2 \setminus \{(0, n) \mid n \in \mathbb{Z}\}$ (integer points on the y-axis are removed) and $Y = \mathbb{R}^2 \setminus \{(0, y) \mid y \in \mathbb{R}\}$ (the y-axis is removed) be subspaces of $\mathbb{R}^2$ with the standard topology. Prove or disprove: There exists a continuous surjection $X \to Y$.

5. Let X and Y be metrizable spaces with metrics d_X and d_Y respectively. Let $f : X \to Y$ be a function. Prove that f is continuous if and only if given $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$.

6. For a surjective map $p : X \to Y$, recall that $Z \subset X$ is *saturated with respect to* p if Z contains every set $p^{-1}(\{y\})$ that it intersects, for any $y \in Y$. Assume that p is a quotient map and let Z be a saturated subspace of X with respect to p. Prove that if p is an open map then the map $q : Z \to f(Z)$, obtained by restricting p, is a quotient map.

7. Let $f : X \to Y$ be a continous function where Y is a Hausdorff space.
   Prove or disprove the following: The graph of f is a closed subspace of $X \times Y$.

Section II: ALGEBRAIC TOPOLOGY

1. Let S^1 be the complex numbers with unit norm. Let $f_n : S^1 \to \mathbb{C} \setminus \{0\}$ be defined by $f(z) = z^n$ for a positive integer n. Prove that f_n is not null-homotopic.

2. Let X be the union of the twelve edges, together with the eight vertices, of the unit cube in 3-space. Determine the fundamental group of X.

3. A space X is said to be contractible if the identity map $i_X : X \to X$ is nulhomotopic. Prove that that a contractible space is path connected.

4. Prove that the n-sphere S^n is simply connected for $n \geq 2$.

5. Let $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ be continuous maps such that $h = g \circ f$. Prove that if g and h are covering maps then f is also a covering map.

6. Let A, B and C be pairwise distinct points of the plane $\mathbb{R}^2$.
   Prove or disprove: $\mathbb{R}^2 \setminus \{A, B, C\}$ is a deformation retract of $\mathbb{R}^2 \setminus \{A, B\}$.

7. Let $p \in S^1$ and let X be the quotient space of the torus $\mathbb{T} = S^1 \times S^1$ by the subspace $S^1 \times \{p\}$ (where $S^1 \times \{p\}$ is identified to a point). Determine all homology groups of the space X.

Department of Mathematics and Statistics
University of South Florida

TOPOLOGY QUALIFYING EXAM
September 24, 2016

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Prove or disprove: Let A, B be subspaces of X, Y, respectively. Then the product topology on $A \times B$ is the same as the subspace topology.

2. Let $X = \mathbb{R}^2 \setminus \{0\}$ and $Y = S^1 \cup ([0,1] \times \{0\}) \subset \mathbb{R}^2$ be the union of the unit circle and the unit interval on the x-axis. Prove or disprove: X is homeomorphic to Y.

3. Outline a proof that any second countable regular space is metrizable.

4. Let A, B, and A_α be subsets of a topological space. Prove or disprove each of the following three equalities:

   (a) $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$.

   (b) $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$.

   (c) $\overline{\cap A_\alpha} = \cap \overline{A_\alpha}$.

5. Let $f : X \to Y$ be a surjective continuous map. Let $X^* = \{f^{-1}(\{y\}), \quad y \in Y\}$ be given the quotient topology. Let $g : X^* \to Y$ be the bijective continuous map induced from f (that is, $f = g \circ p$, where $p : X \to X^*$ is the projection map).

   Prove that the map $g : X^* \to Y$ is a homeomorphism if and only if f is a quotient map.

6. Show that any compact Hausdorff space is normal.

7. Let $f : S^1 \to \mathbb{R}$ be a continuous function from the unit circle to the real line.

   (1) Prove that there exists $z_0 \in S^1$, such that $f(z_0) = f(-z_0)$.

   (2) Determine whether f can be surjective.

Section II: ALGEBRAIC TOPOLOGY

1. Let L_i, $i = 1, 2, 3$, be three paiwise disjount lines in $\mathbb{R}^3$. Determine the fundamental group of $\mathbb{R}^3 \setminus (L_1 \cup L_2 \cup L_3)$.

2. State the equivalence of covering spaces, and classify covering spaces of the circle S^1.

3. Exhibit, with a proof, a space X such that the fundamental group $\pi_1(X)$ and the first homology group $H_1(X)$ are not isomorphic.

4. State the definition of *homotpy type* of a space. Then prove that the space $\mathbb{R}^{m+1} \setminus \{0\}$ and the sphere S^m have the same homotopy type.

5. For $m \geq 3$, prove that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^2$.

6. For a given non-negative integer n, give the list (with an explanation) of all compact connected surfaces X (with or without boundary) such that the first homology group $H_1(X)$ has rank n.

7. Determine the homology groups of the following chain complex, where $C_n = \mathbb{Z}$, ∂_{2n} are multiplication by 2 and ∂_{2n+1} are zero maps for each $n \geq 0$:

$$\cdots \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3 = 0} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1 = 0} \mathbb{Z} \longrightarrow 0.$$

Department of Mathematics and Statistics
University of South Florida

TOPOLOGY QUALIFYING EXAM
January 28, 2017

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Let X be a space satisfying the T_1-axiom, and let A be a subset of X. Prove that $x \in X$ is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

2. Let (X_1, d_1) and (X_2, d_2) be metric spaces with metric topologies. Let $x = (x_1, x_2), y = (y_1, y_2) \in X_1 \times X_2$. Show that $d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$ defines a metric on $X_1 \times X_2$, that induces the product topology.

3. Let A and B be proper subsets of connected spaces X and Y, respectively. Prove that $(X \times Y) \setminus (A \times B)$ is connected.

4. Let Y_i, $i = 1, 2$, be compact Hausdorff spaces with a subspace X such that $Y_i \setminus X$ consists of a single point. Prove that there is a homeomorphism $h : Y_1 \to Y_2$ such that $h|_X$ is the identity map on X.

5. Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. State the definition of uniform convergence for (f_n), and prove that if (f_n) converges uniformly to f, then f is continuous.

6. Let $f : X \to Y$ be a quotient map. Suppose that Y is connected and for every $y \in Y$, $f^{-1}(y)$ is connected. Prove that X is connected.

7. Prove or disprove: every metric space is normal.

Section II: ALGEBRAIC TOPOLOGY

1. Determine the fundamental group of the wedge (one-point union) of two tori, $T \vee T$.
2. Let n be a positive integer. Find all closed (i.e., compact without boundary) surfaces F such that there exists a surjective homomorphism $H_1(F) \to (\mathbb{Z}_2)^n$.
3. Construct a chain complex $\mathcal{C} = (C_n, \partial_n)$ such that $H_0(\mathcal{C}) \cong \mathbb{Z}$, $H_i(\mathcal{C}) \cong \mathbb{Z}_2$ for $i = 1, 2$ and $H_j(\mathcal{C}) = 0$ for $j \neq 0, 1, 2$.
4. Prove or disprove: There exists a continuous map $P \times \cdots \times P \to S^1$ that is not null-homotopic, where P denotes the projective plane, n is a positive integer, $P \times \cdots \times P$ is the n-fold product and S^1 denotes the circle.
5. Let $p : (E, e_0) \to (B, b_0)$ be a covering map. Show that any path in B beginning at b_0 has a unique lifting to a path in E beginning at e_0.
6. State the definition of homotopy equivalence of continuous functions. Then prove that a homotopy equivalence induces an isomorphism on the fundamental groups.
7. Let $p : (E, e_0) \to (B, b_0)$ be a covering map and assume that E is simply connected. Prove that the group of covering transformations of (E, p, B) is isomorphic to the fundamental group of B.

Department of Mathematics and Statistics
University of South Florida

TOPOLOGY QUALIFYING EXAM
May 13, 2017

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Let (X, d) be a metric space. Let $B = \overline{B}_d(x, r) = \{y \in X \mid d(x, y) \leq r\}$ be a closed ball, where $x \in X$, $r > 0$.

   Prove or disprove: For any subset A of X, $\mathrm{C}\ell_B(A \cap B)$ is compact, where $\mathrm{C}\ell_B(C)$ denotes the closure of C in B.

2. Recall that a space is Lindelöf if every open covering contains a countable subcover.

   (a) Prove that a space is Lindelöf if and only if every covering by basis elements contains a countable subcover.

   (b) Prove that the second countability implies Lindelöf.

3. Prove or disprove: If there exists a continuous bijection between two topological spaces, then they are homeomorphic.

4. Show that if Y is compact, then the first projection $p_1 : X \times Y \to X$ is a closed map.

5. Give an example, with proofs, of connected space that is not path-connected.

6. Let $f : X \to Y$ be a continuous map, where Y is an ordered set with the order topology. Prove that if X is compact, then there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

7. Prove that every regular space with a countable basis is normal.

Section II: ALGEBRAIC TOPOLOGY

1. Determine, with a proof, the fundamental group of the wedge (one-point union) of two projective planes, $\mathbb{P}^2 \vee \mathbb{P}^2$.

2. Let K be the Klein bottle, and $X = K\#K\#K$. Find all pairs of closed surfaces F_1 and F_2 such that $F_1\#F_2$ is homeomorphic to X. Specify F_i, $i = 1, 2$, by orientability and genus, and provide a proof.

3. Let $\{C_n, \partial_n\}$, $n \in \mathbb{Z}_{\geq 0}$, be a chain complex such that $C_1 = \mathbb{Z}$, $C_2 = \mathbb{Z} \oplus \mathbb{Z}$ and $C_n = 0$ for $n \neq 1, 2$, where $\partial_2(m, n) = 2m + 2n$ for $(m, n) \in C_2$. Determine all homology groups.

4. Prove or disprove: There exists a retraction of a torus T onto a subspace of T homeomorphic to the bouquet of three circles.

5. A space is called *contractible* if the identity map $i : X \to X$ is nulhomotopic. Prove that a retract of a contractible space is contractible.

6. Show that if $n > 1$, then any continuous map $f : S^n \to T$ from the n-sphere to the torus is nullhomotopic.

7. Assume the theorem that if $h : S^1 \to S^1$ is a continuous antipode-preserving map ($h(-x) = -h(x)$ for all $x \in S^1$), then h is not nullhomotopic.

   (a) Show that there is no continuous antipode-preserving map $g : S^2 \to S^1$.

   (b) Show that for any continuous map $f : S^2 \to \mathbb{R}^2$, there exists $x \in S^2$ such that $f(x) = f(-x)$.

Department of Mathematics and Statistics
University of South Florida

TOPOLOGY QUALIFYING EXAM
September 23, 2017

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Prove that if A and B are disjoint compact subspaces of the Hausdorff space X, then there exists disjoint open sets U and V containing A and B, respectively.

2. Recall that a space X is called *completely regular* if one-point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

   Prove that a product (of any cardinality) of completely regular spaces is completely regular.

3. Prove that the components of a space X are connected disjoint subspaces whose union is X, such that each nonempty connected subspace of X intersects only one of them.

4. Let X be a topological space and $A \subset X$. Prove that if there is a sequence of points of A converging to x, then $x \in \overline{A}$. Prove also that the converse holds if X is first countable.

5. Prove that a space X is Hausdorff *if and only if* the diagonal $\{(x, x) \mid x \in X\}$ is closed in $X \times X$.

6. Let $\mathbb{R}^\omega$ be the Euclidean space of countably infinite dimension. Consider $\mathbb{R}^\omega$ with the box topology. Prove or disprove that the function f from $\mathbb{R}$ to $\mathbb{R}^\omega$ is continuous, where
$$f(x) = (x, \frac{x}{2}, \frac{x}{3}, \ldots).$$

7. Prove that $\mathbb{R}^\omega$ with product topology is connected.

Section II: ALGEBRAIC TOPOLOGY

1. Let $T = S^1 \times S^1$ be a torus and $x \in T$. Prove or disprove: There exists a continuous surjective map $f : T \to T$ such that the induced homomorphism $f_* : H_1(T, x) \to H_1(T, x)$ is the zero-map (meaning that the image is $\{0\}$).
2. Prove that the free group of rank 2 contains the free group of rank n as a subgroup for any positive integer $n > 2$.
3. Let X be the union of two copies of the 3-sphere S^3 having a single point in common. Compute the fundamental group of the space X.
4. Prove that if $f : S^1 \to S^1$ is nullhomotopic then f has a fixed point and f maps some x to its antipode $-x$.
5. Find a covering space of the torus $S^1 \times S^1$ corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by the element $m \times 0$, where m is a positive integer.
6. Determine the homology groups of the Klein bottle.
7. Let T be the torus and K be the Klein bottle. Find all pairs (S_1, S_2) of closed (i.e., compact without boundary) surfaces S_1 and S_2 such that $T \# S_1$ is homeomorphic to $K \# S_2$ and the genus of the resulting surface is less than 7.

Department of Mathematics and Statistics
University of South Florida

TOPOLOGY QUALIFYING EXAM
January 27, 2018

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Suppose that there are embeddings $f : X \to Y$ and $g : Y \to X$. Prove or disprove: X and Y are homeomorphic.

2. Recall that X is *locally compact* if for each point $x \in X$, there is a compact subspace that contains a neighborhood of x.

   Prove that if there is a compact Hausdorff space Y containing X as a subspace such that $Y \setminus X$ is a single point, then X is locally compact and Hausdorff.

3. Prove or disprove: Every connected regular space with a countable basis is uncountable.

4. Let A and B be subspaces of X and Y, respectively. Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

5. Prove that if $f, g : X \to \mathbb{R}$ are continuous functions, then $f + g$ is also continuous.

6. Consider the projection $p_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ onto the first coordinate, $p_1(x, y) = x$.

   (1) Prove or disprove: p_1 is an open map.

   (2) Prove or disprove: p_1 is a closed map.

7. Prove or disprove: An injective continuous function is an embedding.

Section II: ALGEBRAIC TOPOLOGY

1. List all subgroups G of $\pi_1(S^1, b_0)$, where S^1 is the circle and $b_0 \in S^1$. For each G, determine whether there exists a space X and a continuous map $f : X \to S^1$ such that the image of the induced homomorphism f_* is G.

2. Let $T = S^1 \times S^1$ be a torus and B be the bouquet of two circles. Prove or disprove: there is a continuous surjective map $f : T \to B$ that induces a surjective homomorphism on the fundamental groups.

3. Let $T_m = \#_m T$ and $K_n = \#_n K$ be the m, n-fold connected sum of the torus ($T = S^1 \times S^1$) and the Klein bottle, respectively. Find all possible non-negative integers m_i, n_i, $i = 1, 2$ such that $T_{m_1} \# K_{n_1}$ is homeomorphic to $T_{m_2} \# K_{n_2}$.

4. State and prove the Brouwer fixed-point theorem for the disc.

5. Let X be the space obtained from $\mathbb{R}^3$ by removing the three (x-, y- and z-) axes. Determine the fundamental group of X.

6. Let $p : E \to B$ be a covering map. Prove or disprove:
   (1) If E_0 is a subspace of E, then $p|_{E_0} : E_0 \to p(E_0)$ is a covering map.
   (2) If B_0 is a subspace of B, and $E_0 = p^{-1}(B_0)$, then $p|_{E_0} : E_0 \to B_0$ is a covering map.

7. Determine all homology groups of the space $S^2 \cup \{(x, y, z) \mid -1 \leq x \leq 1,\ y = z = 0\}$.