

Department of Mathematics and Statistics
University of South Florida

TOPOLOGY QUALIFYING EXAM

January 23, 2016

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. (1) Define a subbasis \mathcal{S} and a basis \mathcal{B} for a topology on X , (2) define the topology generated by \mathcal{S} , (3) prove that it is indeed a topology, and (4) give an example, with a proof, of a subbasis that generates the standard topology of \mathbb{R}^2 and that is not a basis.
2. Let Y be a subspace of X . Prove that a pair of disjoint nonempty sets A and B of Y forms a separation of Y if and only if $Y = A \cup B$ and neither of A nor B contains a limit point of the other.
3. State the definitions of a space being (1) second countable, (2) Lindelöf, and (3) separable. Show that the second countability implies the other two.
4. Let $A \subset X$; let $f : A \rightarrow Y$ be a continuous map with Y being Hausdorff. Show that if f may be extended to a continuous map $g : \overline{A} \rightarrow Y$, then g is uniquely determined by f .
5. Recall that a collection \mathcal{C} of subsets of X is said to have the *finite intersection property* if for every finite subcollection $\{C_1, \dots, C_n\}$ of \mathcal{C} , the intersection $C_1 \cap \dots \cap C_n$ is non-empty.

State a necessary and sufficient condition, with a proof, of X being compact using the finite intersection property of closed subsets of X .

6. Prove or disprove: Every compact subspace of a normal space is normal.
7. Recall that the square metric on \mathbb{R}^n is defined by

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

for $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

Prove that topologies on \mathbb{R}^n induced by the square metric and the euclidian metric are the same as the product topology.

Section II: ALGEBRAIC TOPOLOGY

1. Let $p : E \rightarrow B$ be a covering map. Let e_0, e_1 be points in E such that $p(e_0) = p(e_1) = b_0 \in B$ and let $H_i = p_*(\pi_1(E, e_i))$ for $i = 0, 1$. Prove that H_0 and H_1 are conjugate subgroups of $\pi_1(B, b_0)$.
2. Determine the homology groups of the union of the unit 2-sphere $S^2 \subset \mathbb{R}^3$ and its equatorial disk $D = \{(x, y, z) \mid z = 0, x^2 + y^2 \leq 1\}$.
3. Let $n > 1$ be a positive integer. Provide two spaces X, Y , and a continuous map $f : X \rightarrow Y$ such that the induced map on the first homology groups $f_* : H_1(X) \rightarrow H_1(Y)$ is an epimorphism $\mathbb{Z} \rightarrow \mathbb{Z}_n$.
4. Define a covering space and a local homeomorphism. Then provide, with a proof, a local homeomorphism that is not a covering space.
5. Let X be the space obtained by pasting the edges of a polygonal region with the labeling scheme $abcdabcd$.
 - (1) Compute the first homology group $H_1(X)$.
 - (2) Identify X with one of the surfaces in the list of the Classification Theorem.
6. Determine the fundamental group of the quotient space obtained from the real projective plane \mathbb{P} by identifying three distinct points on \mathbb{P} .
7. Prove or disprove: the bouquet of two circles can be embedded in the solid torus $S^1 \times D^2$ as a retract.

Department of Mathematics and Statistics
University of South Florida

TOPOLOGY QUALIFYING EXAM

May 14, 2016

Examiners: Dr. M. Elhamedi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Let $K = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Let $\mathcal{B}_K = \{(a, b), (a, b) \setminus K \mid a, b \in \mathbb{R}, a < b\}$. Show that \mathcal{B}_K forms a basis of a topology, \mathbb{R}_K , on \mathbb{R} , and that \mathbb{R}_K is strictly finer than the standard topology of \mathbb{R} .
2. Recall that a space X is *countably compact* if any countable open covering of X contains a finite subcovering. Prove that X is countably compact if and only if every nested sequence $C_1 \supset C_2 \supset \cdots$ of closed nonempty subsets of X has a nonempty intersection.
3. Assume that all singletons are closed in X . Show that if X is regular, then every pair of points of X has neighborhoods whose closures are disjoint.
4. Let $X = \mathbb{R}^2 \setminus \{(0, n) \mid n \in \mathbb{Z}\}$ (integer points on the y -axis are removed) and $Y = \mathbb{R}^2 \setminus \{(0, y) \mid y \in \mathbb{R}\}$ (the y -axis is removed) be subspaces of \mathbb{R}^2 with the standard topology. Prove or disprove: There exists a continuous surjection $X \rightarrow Y$.
5. Let X and Y be metrizable spaces with metrics d_X and d_Y respectively. Let $f : X \rightarrow Y$ be a function. Prove that f is continuous if and only if given $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$.
6. For a surjective map $p : X \rightarrow Y$, recall that $Z \subset X$ is *saturated with respect to p* if Z contains every set $p^{-1}(\{y\})$ that it intersects, for any $y \in Y$. Assume that p is a quotient map and let Z be a saturated subspace of X with respect to p . Prove that if p is an open map then the map $q : Z \rightarrow f(Z)$, obtained by restricting p , is a quotient map.
7. Let $f : X \rightarrow Y$ be a continuous function where Y is a Hausdorff space. Prove or disprove the following: The graph of f is a closed subspace of $X \times Y$.

Section II: ALGEBRAIC TOPOLOGY

1. Let S^1 be the complex numbers with unit norm. Let $f_n : S^1 \rightarrow \mathbb{C} \setminus \{0\}$ be defined by $f(z) = z^n$ for a positive integer n . Prove that f_n is not null-homotopic.
2. Let X be the union of the twelve edges, together with the eight vertices, of the unit cube in 3-space. Determine the fundamental group of X .
3. A space X is said to be contractible if the identity map $i_X : X \rightarrow X$ is nullhomotopic. Prove that a contractible space is path connected.
4. Prove that the n -sphere S^n is simply connected for $n \geq 2$.
5. Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : X \rightarrow Z$ be continuous maps such that $h = g \circ f$. Prove that if g and h are covering maps then f is also a covering map.
6. Let A, B and C be pairwise distinct points of the plane \mathbb{R}^2 .
Prove or disprove: $\mathbb{R}^2 \setminus \{A, B, C\}$ is a deformation retract of $\mathbb{R}^2 \setminus \{A, B\}$.
7. Let $p \in S^1$ and let X be the quotient space of the torus $\mathbb{T} = S^1 \times S^1$ by the subspace $S^1 \times \{p\}$ (where $S^1 \times \{p\}$ is identified to a point). Determine all homology groups of the space X .

Department of Mathematics and Statistics
University of South Florida
TOPOLOGY QUALIFYING EXAM
September 24, 2016

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Prove or disprove: Let A, B be subspaces of X, Y , respectively. Then the product topology on $A \times B$ is the same as the subspace topology.
2. Let $X = \mathbb{R}^2 \setminus \{0\}$ and $Y = S^1 \cup ([0, 1] \times \{0\}) \subset \mathbb{R}^2$ be the union of the unit circle and the unit interval on the x -axis. Prove or disprove: X is homeomorphic to Y .
3. Outline a proof that any second countable regular space is metrizable.
4. Let A, B , and A_α be subsets of a topological space. Prove or disprove each of the following three equalities:

(a) $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$.

(b) $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$.

(c) $\overline{\cap A_\alpha} = \cap \overline{A_\alpha}$.

5. Let $f : X \rightarrow Y$ be a surjective continuous map. Let $X^* = \{f^{-1}(\{y\}), y \in Y\}$ be given the quotient topology. Let $g : X^* \rightarrow Y$ be the bijective continuous map induced from f (that is, $f = g \circ p$, where $p : X \rightarrow X^*$ is the projection map).

Prove that the map $g : X^* \rightarrow Y$ is a homeomorphism if and only if f is a quotient map.

6. Show that any compact Hausdorff space is normal.
7. Let $f : S^1 \rightarrow \mathbb{R}$ be a continuous function from the unit circle to the real line.
 - (1) Prove that there exists $z_0 \in S^1$, such that $f(z_0) = f(-z_0)$.
 - (2) Determine whether f can be surjective.

Section II: ALGEBRAIC TOPOLOGY

1. Let L_i , $i = 1, 2, 3$, be three pairwise disjoint lines in \mathbb{R}^3 . Determine the fundamental group of $\mathbb{R}^3 \setminus (L_1 \cup L_2 \cup L_3)$.
2. State the equivalence of covering spaces, and classify covering spaces of the circle S^1 .
3. Exhibit, with a proof, a space X such that the fundamental group $\pi_1(X)$ and the first homology group $H_1(X)$ are not isomorphic.
4. State the definition of *homotopy type* of a space. Then prove that the space $\mathbb{R}^{m+1} \setminus \{0\}$ and the sphere S^m have the same homotopy type.
5. For $m \geq 3$, prove that \mathbb{R}^m is not homeomorphic to \mathbb{R}^2 .
6. For a given non-negative integer n , give the list (with an explanation) of all compact connected surfaces X (with or without boundary) such that the first homology group $H_1(X)$ has rank n .
7. Determine the homology groups of the following chain complex, where $C_n = \mathbb{Z}$, ∂_{2n} are multiplication by 2 and ∂_{2n+1} are zero maps for each $n \geq 0$:

$$\dots \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3=0} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1=0} \mathbb{Z} \longrightarrow 0.$$

Department of Mathematics and Statistics
University of South Florida

TOPOLOGY QUALIFYING EXAM

January 28, 2017

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Let X be a space satisfying the T_1 -axiom, and let A be a subset of X . Prove that $x \in X$ is a limit point of A if and only if every neighborhood of x contains infinitely many points of A .
2. Let (X_1, d_1) and (X_2, d_2) be metric spaces with metric topologies. Let $x = (x_1, x_2), y = (y_1, y_2) \in X_1 \times X_2$. Show that $d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$ defines a metric on $X_1 \times X_2$, that induces the product topology.
3. Let A and B be proper subsets of connected spaces X and Y , respectively. Prove that $(X \times Y) \setminus (A \times B)$ is connected.
4. Let $Y_i, i = 1, 2$, be compact Hausdorff spaces with a subspace X such that $Y_i \setminus X$ consists of a single point. Prove that there is a homeomorphism $h : Y_1 \rightarrow Y_2$ such that $h|_X$ is the identity map on X .
5. Let $f_n : X \rightarrow Y$ be a sequence of continuous functions from the topological space X to the metric space Y . State the definition of uniform convergence for (f_n) , and prove that if (f_n) converges uniformly to f , then f is continuous.
6. Let $f : X \rightarrow Y$ be a quotient map. Suppose that Y is connected and for every $y \in Y$, $f^{-1}(y)$ is connected. Prove that X is connected.
7. Prove or disprove: every metric space is normal.

Section II: ALGEBRAIC TOPOLOGY

1. Determine the fundamental group of the wedge (one-point union) of two tori, $T \vee T$.
2. Let n be a positive integer. Find all closed (i.e., compact without boundary) surfaces F such that there exists a surjective homomorphism $H_1(F) \rightarrow (\mathbb{Z}_2)^n$.
3. Construct a chain complex $\mathcal{C} = (C_n, \partial_n)$ such that $H_0(\mathcal{C}) \cong \mathbb{Z}$, $H_i(\mathcal{C}) \cong \mathbb{Z}_2$ for $i = 1, 2$ and $H_j(\mathcal{C}) = 0$ for $j \neq 0, 1, 2$.
4. Prove or disprove: There exists a continuous map $P \times \cdots \times P \rightarrow S^1$ that is not null-homotopic, where P denotes the projective plane, n is a positive integer, $P \times \cdots \times P$ is the n -fold product and S^1 denotes the circle.
5. Let $p : (E, e_0) \rightarrow (B, b_0)$ be a covering map. Show that any path in B beginning at b_0 has a unique lifting to a path in E beginning at e_0 .
6. State the definition of homotopy equivalence of continuous functions. Then prove that a homotopy equivalence induces an isomorphism on the fundamental groups.
7. Let $p : (E, e_0) \rightarrow (B, b_0)$ be a covering map and assume that E is simply connected. Prove that the group of covering transformations of (E, p, B) is isomorphic to the fundamental group of B .

Department of Mathematics and Statistics
University of South Florida
TOPOLOGY QUALIFYING EXAM
May 13, 2017

Examiners: Dr. M. Elhamdadi, Dr. M. Saito

Instructions: For Ph.D. level, complete at least seven problems, at least three problems from each section. For Master's level, complete at least five problems, at least one problem from each section. **State all theorems or lemmas you use.**

Section I: POINT SET TOPOLOGY

1. Let (X, d) be a metric space. Let $B = \overline{B}_d(x, r) = \{y \in X \mid d(x, y) \leq r\}$ be a closed ball, where $x \in X$, $r > 0$.
Prove or disprove: For any subset A of X , $Cl_B(A \cap B)$ is compact, where $Cl_B(C)$ denotes the closure of C in B .
2. Recall that a space is Lindelöf if every open covering contains a countable subcover.
 - (a) Prove that a space is Lindelöf if and only if every covering by basis elements contains a countable subcover.
 - (b) Prove that the second countability implies Lindelöf.
3. Prove or disprove: If there exists a continuous bijection between two topological spaces, then they are homeomorphic.
4. Show that if Y is compact, then the first projection $p_1 : X \times Y \rightarrow X$ is a closed map.
5. Give an example, with proofs, of connected space that is not path-connected.
6. Let $f : X \rightarrow Y$ be a continuous map, where Y is an ordered set with the order topology. Prove that if X is compact, then there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.
7. Prove that every regular space with a countable basis is normal.

Section II: ALGEBRAIC TOPOLOGY

1. Determine, with a proof, the fundamental group of the wedge (one-point union) of two projective planes, $\mathbb{P}^2 \vee \mathbb{P}^2$.
2. Let K be the Klein bottle, and $X = K \# K \# K$. Find all pairs of closed surfaces F_1 and F_2 such that $F_1 \# F_2$ is homeomorphic to X . Specify F_i , $i = 1, 2$, by orientability and genus, and provide a proof.
3. Let $\{C_n, \partial_n\}$, $n \in \mathbb{Z}_{\geq 0}$, be a chain complex such that $C_1 = \mathbb{Z}$, $C_2 = \mathbb{Z} \oplus \mathbb{Z}$ and $C_n = 0$ for $n \neq 1, 2$, where $\partial_2(m, n) = 2m + 2n$ for $(m, n) \in C_2$. Determine all homology groups.
4. Prove or disprove: There exists a retraction of a torus T onto a subspace of T homeomorphic to the bouquet of three circles.
5. A space is called *contractible* if the identity map $i : X \rightarrow X$ is nullhomotopic. Prove that a retract of a contractible space is contractible.
6. Show that if $n > 1$, then any continuous map $f : S^n \rightarrow T$ from the n -sphere to the torus is nullhomotopic.
7. Assume the theorem that if $h : S^1 \rightarrow S^1$ is a continuous antipode-preserving map ($h(-x) = -h(x)$ for all $x \in S^1$), then h is not nullhomotopic.
 - (a) Show that there is no continuous antipode-preserving map $g : S^2 \rightarrow S^1$.
 - (b) Show that for any continuous map $f : S^2 \rightarrow \mathbb{R}^2$, there exists $x \in S^2$ such that $f(x) = f(-x)$.